UNIVERSITY OF ILLINOIS

COLLEGE OF ENGINEERING

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BULLETIN 485

MINIMUM WEIGHT OF PLASTICALLY DESIGNED STEEL FRAMES

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ABSTRACT

A METHOD IS DEVELOPED FOR THE MINIMUM-WEIGHT DESIGN OF STEEL STRUCTURES BASED ON PLASTIC ANALYSIS. THE PROCEDURE INCLUDES THE EFFECTS OF AXIAL LOADING, OVERALL FRAME INSTABILITY DUE TO SIDESWAY, AND THE NONLINEAR RELATIONSHIP BETWEEN WEIGHT AND MOMENT CAPACITY OF STANDARD SECTIONS.

A PROGRAM, WHICH WAS WRITTEN FOR THE 7090 COMPUTER, WILL HANDLE GABLE AND OTHER NONORTHOGONAL FRAMES AS WELL AS RECTANGULAR FRAMES.

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I. INTRODUCTION

A. GENERAL REMARKS

Normally a structure is designed to serve certain functional requirements with an adequate factor of safety against failure and, usually, with a minimum of cost. Functional requirements determine in large degree the geometry and loads. Factors of safety, adequate or otherwise, are usually prescribed by codes and ordinances.

While the determination of the maximum load-carrying capacity of a given frame is a problem for which only one answer exists, many feasible designs for a given geometry and loading may exist. Only one solution, however, provides the minimum-weight design.

The determinate structure of given geometry can be designed for minimum weight without resorting to a trial procedure, because the distribution over the structure of internal moments and forces is not dependent on member sizes. On the other hand, the indeterminate structure, when designed on the basis of elastic analysis, requires a method of trial to approach the minimum-weight design. This fact is manifested by the presence of the stiffness or flexibility factor in the matrix of coefficients which relate the redundant moments in an indeterminate problem. Before an elastic analysis can be made, therefore, an estimate of the individual sizes must be made. Determination of the redundant moments usually indicates that different member sizes are required, and that further analysis is necessary.

With the development of the ultimate load method of analysis, however, it is possible to determine an admissible distribution of moments over an indeterminate structure without a previous estimate of the member sizes. This simplification is possible because the matrix of coefficients which relate the critical moments are functions of the geometry of the structure and of the hinge positions, and are independent of member sizes.

Since member sizes do not have to be estimated before an analysis is made, it should be possible to determine a distribution of moments for a given structure which will yield a minimum-weight solution. In order that this may be accomplished, suitable relationships between weight of member and moment capacity of both beams and columns must be established. In addition, a method of proceeding efficiently from one distribution of moments to another which will yield a lighter structure must be developed.

B. PREVIOUS WORK

A number of researchers have interested themselves in the minimum-weight design of steel structures designed by the "plastic" or "ultimate load" method.(3) (6) (7) (8) (11) (12) (14) (16) (19) (20) (24)* For the most

*Numbers in parentheses refer to entries in the Bibliography.

part, the approach has been of a theoretical nature in that a continuous spectrum of structural shapes is assumed available. Many of the researchers have further limited the problem by specifying a linear relationship between the plastic moment capacities and weights of this continuous spectrum of shapes. Others have assumed a nonlinear relationship of the type $g = K(M_p)^n$, where g is the weight per foot of the section, M_p is the plastic moment of the section, and K and n are constants. In none of the papers reviewed has the investigator considered the reduction in plastic moment capacity in the presence of axial force or the stability of the beam-column as they affect the minimum-weight problem. In any case, when the designer attempts to apply these methods to the minimum-weight design of structures other than very simple ones, he is faced with a formidable array of calculations which may or may not lead to the minimum-weight design of a real structure. Since the majority of structures for which plastic analysis is presently considered to be appropriate are constructed of available standard structural shapes, a method which purports to be useful to the designer must, in its final application, relate the minimum-weight problem to the properties of these shapes rather than to a continuous spectrum of shapes. This requirement assumes even greater significance when one notes the relationships among properties of those standard shapes which are important to minimum weight. These relationships do not plot as well-defined curves, but exhibit some scatter.

At present, the designer must resort to a comparison between the weights of succeeding trial designs in an attempt to arrive at a minimum-weight solution. He has no assurance of having found a minimum-weight design when he finally accepts his best solution, nor has he a way of knowing how close he is to the minimum-weight design. Even if an electronic computer is used to make comparisons of trial designs for a structure consisting of, say, ten or more members, the time involved in considering all possible solutions becomes prohibitive.

C. OBJECTIVE AND SCOPE

The objective of this study is to develop a method for the minimum-weight design of steel structures, based on plastic analysis, which satisfies the following requirements:

(1) The method will embrace the problems of axial compression as well as flexural loading, lateral displacement (sidesway) of the structure, and the nonlinear relationship between unit weight of members and their moment capacities.

(2) The method will allow the determination of minimum weight for the structure which is designed for standard structural shapes as well as the structure for which a continuous spectrum of shapes may be available.

(3) The method will converge to the minimum-weight design by consideration of only a small percentage of the feasible solutions.

D. NOTATION

A	= augmented matrix of the array of linear restrictions
a_i	= constant (Figure 7)
a_i'	= constant (Figure 7)
a_{ij}	= coefficients in the array of linear restrictions
a_{ijk}	= coefficients in the array of basic equations for equation generation
a_k	= constant (Figure 6)
B	= constant for a given L/r_x (Equation 3.5)
b_i	= constant (Figure 7)

b_i' = constant (Figure 7)

b_k = constant (Figure 6)

c_i = length of i^{th} column in feet

c_j = coefficients of variables in the objective function

c_k = length of k^{th} beam in feet

D_i = weight per foot of i^{th} column

D_i' = weight per foot of i^{th} column when subject to sidesway

D_k = weight per foot of k^{th} beam

d_j = constant corresponding to variables in array of linear restrictions each of which limits a single variable

F_j = sum of products of coefficients of solution basis and corresponding scalar multipliers in j^{th} column of tableau

f = objective function

F_w = frame weight

F_w' = frame weight minus a constant

G = constant for a given L/r_x (Equation 3.6)

L = column length

M_p = pure-bending fully plastic moment

M_0 = moment capacity of a column of given length in conjunction with a given axial load

m = number of linear restrictions expressed by Equation 2.5

n = number of structural variables

n_b = number of beams in frame

n_c = number of columns in frame

P = applied axial load

$\overline{P}_i$ = i^{th} vector in solution

$\overline{P}_j$ = j^{th} column vector in the array of linear restrictions

$\overline{P}_0$ = column vector of elements representing the external work for the corresponding restrictions

P_y = product of the cross sectional area and the yield stress of the steel

p = number of beam equations

q = number of panel equations

r = number of joint equations

r_x = radius of gyration with respect to x-x axis

X_{ij} = scalar multipliers of the vectors in solution

X_j = structural variables

X_{n+i} = slack variables Equation 2.5

X_{n+m+j} = slack variables Equation 2.6

β = mechanism angle

θ = criterion by which vector going out of solution is determined

Φ = minimum-weight solution consisting of n sections for the n members in the frame

II. METHOD OF SOLUTION

A. PLASTIC DESIGN

One of the assumptions underlying plastic design of steel structures is that the ultimate moment-resisting capacity at a cross section is reached when the stress distribution in pure bending is that shown in Figure 1. Since this distribution represents complete plastification of the cross section, rotation of the member at a point where such a distribution develops proceeds with no increase in moment, provided, as is customary, that strain hardening is neglected. In effect, then, with respect to loading beyond that which produces a fully plastic moment at some cross section, a structural member behaves as though there were a hinge at that cross section. Such a fictitious hinge is called a plastic hinge. The corresponding moment-rotation diagram is shown in Figure 2.

The formation of one plastic hinge in a statically determinate structure results in complete collapse: i.e., the ultimate load has been reached. For complete collapse of a statically indeterminate structure, $N + 1$ plastic hinges are required, where N is the degree of indeterminacy. Partial collapse may result from the formation of only one hinge. This would occur, for example, in a member cantilevered from the remainder of the structure. It is possible, then, for partial collapse of an indeterminate structure to result from the formation of from one to N plastic hinges. In addition, an indeterminate structure will exhibit a number of different modes of collapse, the critical mode being that mode which results in the least value of the ultimate load. The system that results when the structure is in a state of collapse is called a mechanism.

An equilibrium equation involving the external loads and the moments at the hinges defining the collapse mechanism may be obtained for each mode of collapse by using the principle of virtual displacements. There will be as many such equations as there are modes of collapse. In the general case, only one of these equilibrium equations represents a safe design, i.e., only one equilibrium equation will correspond to a distribution of moments throughout the structure for which nowhere in a (prismatic) member will the moment exceed the plastic moment of resistance prescribed for that member.

Since only one of the n equations of equilibrium corresponding to the n mechanisms of collapse represents a safe design, all equilibrium equations other than that one may be written as inequalities in which the work done by the internal moments during a virtual displacement exceeds the work done by the external loads. Consideration of the inequalities representing all possible modes of collapse will satisfy all the criteria which are necessary and sufficient to determine the solution. These criteria are:

(1) A collapse mechanism must be obtained.

(2) The structure must be in equilibrium at the collapse load.

(3) The moment at any point in a prismatic member may not exceed the ultimate moment capacity of that member. This criterion is called the yield criterion.

Since each inequality represents a particular mode of collapse, the first criterion is satisfied. Both the second and third conditions are satisfied, provided all possible collapse mechanisms are considered, because an equilibrium equation is obtained if the inequality becomes an equality. This condition can result only if the moments at the mechanism hinges are reduced.

B. THE MINIMIZATION PROBLEM

For a given load system and structure geometry, many feasible designs may be determined. Mathematically this fact is expressed by the existence of more unknowns than equations which relate the unknowns. Furthermore, the distribution of moments over the structure at the ultimate load is influenced by the relative moment capacities of the various members. This fact becomes evident if the structure is observed during the last stages of loading leading to the ultimate load. As the ultimate load is approached, each succeeding hinge brings about a redistribution of moment which differs from the moment distribution that would have prevailed had the structure remained elastic.

It is not feasible to determine the minimum-weight solution for a frame of any size strictly by trial even on a high-speed electronic computer. Using a trial procedure, the 7090 I.B.M. computer required twenty-seven minutes to obtain the minimum-weight solution for a five-member, two-bay, one-story frame. Over 60,000 designs were considered. If one more member had been added to this frame the time required for solution would have been increased by a factor of about twenty. At present this five-member frame, then, represents about the upper limit for trial procedures.

The desired method should proceed from the first solution to the minimum-weight solution with the consideration of only a very small percentage of the possible solutions. Furthermore, it should proceed from one solution to the next without having to restart the solution process. Finally, a criterion to identify the minimum-weight solution must be available. Such a method exists. This method, known as linear programming,[10] [21] [23] was first developed by George B. Dantzig, Marshall Wood, and their associates. Use of the linear programming method for the solution of the minimum-weight problem has been suggested by Charnes and Greenberg.[3]

C. LINEAR PROGRAMMING

A function may be either maximized or minimized by the linear programming method. Since we are concerned with minimization of weight, only minimization is considered. The method described is known as the Simplex Method.

Let it be required to minimize

$$f = \sum_{j=1}^{n} c_j X_j \tag{2.1}$$

subject to

$$\sum_{j=1}^{n} a_{ij} X_j \geq b_i \qquad i = 1,2,\ldots..m \tag{2.2}$$

$$X_j \geq d_j \qquad j = 1,2,\ldots..n \tag{2.3}$$

where $d_j \geq 0$

Equation (2.1) is known as the objective function, and Equations (2.2) and (2.3) are the linear restrictions or side conditions. The simplex method requires both the objective function and the side conditions to be linear. In order to apply formal systematic solution procedures to this problem the above inequalities must be expressed as equalities as follows:

Minimize

$$f = \sum_{j=1}^{n} c_j X_m + \sum_{i=1}^{m} 0X_{n+i} + \sum_{j=1}^{n} 0X_{n+m+j} \qquad (2.4)$$

subject to

$$\sum_{j=1}^{n} a_{ij} X_j - X_{n+i} = b_i \quad i = 1,2,\ldots.m \qquad (2.5)$$

$$X_j - X_{n+m+j} = d_j \quad j = 1,2,\ldots.n \qquad (2.6)$$

where

$d_j \geq 0$

$X_{n+i} \geq 0$

$X_{n+m+j} \geq 0$

X_j = structural variables

X_{n+i} and X_{n+m+j} = surplus variables

a_{ij}, b_i, c_j, d_j = constants

n = number of structural variables

m = number of linear restrictions expressed by Equation (2.5)

The dimensions of the augmented matrix are (m+n) x (m+2n). Although these dimensions define a matrix of (m+n) equations for (m+2n) variables, there is a condition under which all these equations and variables need not be formally stated even though they will always exist. The simplex method guarantees that all solutions are composed of only non-negative values for all variables. Therefore, if $d_j = 0$, Equation (2.6) for the jth variable is redundant and need not be included in the linear restrictions. For this reason, the number of equations that need be stated will vary from m to m+n, and the number of variables that need be considered will vary from m+n to m+2n. However, in the discussions that follow it will be considered that the augmented matrix consists of m+n equations and m+2n variables.

An example will illustrate the above formulation and the subsequent solution.

Minimize

$$f = 2X_1 + 3X_2 \qquad (2.7)$$

subject to

$$\left.\begin{aligned} X_1 + X_2 &\geq 6 \\ X_1 + 2X_2 &\geq 8 \end{aligned}\right\} \qquad (2.8)$$

$$\left.\begin{aligned} \text{where } X_1 &\geq 0 \\ \text{and } \quad X_2 &\geq 0 \end{aligned}\right\} \qquad (2.9)$$

Expressing Equation (2.8) as equalities of the form of Equation (2.5), Equations (2.7) and (2.8) become

$$f = 2X_1 + 3X_2 + 0X_3 + 0X_4 \qquad (2.10)$$

$$X_1 + X_2 - X_3 \quad = 6 \qquad (2.11)$$

$$X_1 + 2X_2 \quad - X_4 = 8 \qquad (2.12)$$

where $X_3 \geq 0$

and $X_4 \geq 0$

In this case it is not necessary to write Equations (2.9) as equalities, because the lower limits for the variables X_1 and X_2 are zero, as was discussed above.

Equations (2.11) and (2.12) may be represented geometrically in two-dimensional space as shown in Figure 3. Lines AB and CD represent Equations (2.12) and (2.11), respectively, for the case $X_3 = X_4 = 0$. Line EF represents one possible location for the objective function. Any point to the right of the vertical axis, above and to the right of line AOD, and above the horizontal axis, represents a feasible or admissible solution which satisfies Equations (2.11) and (2.12). This space is known as the solution space and is shown shaded in Figure 3. Solutions involving negative variables, which would be represented by points to the left of the vertical axis and/or below the horizontal axis, are not admissible in the simplex procedure. In order that the procedure yield a minimum solution the solution space must be a convex set. A convex set exists if lines drawn between all pairs of arbitrarily selected points within the solution space are entirely contained within the solution space. The apexes on the boundary of the solution space are known as extreme points. These extreme points represent solutions which are defined as basic feasible solutions. Points A, O, and D in Figure 3 represent basic feasible solutions. It has been shown that the minimum solution will always be a basic feasible solution.[10]

Primarily, the object of the simplex procedure is to generate a new basic feasible solution from the preceding one such that the new solution is always closer to the minimum solution. Further, the minimum solution must be recognized when it has been reached. This operation may be visualized as the sweeping out of the solution space in Figure 3 by the objective function.

If the augmented matrix is designated $[A]$, Equations (2.4), (2.5), and (2.6) may be expressed in matrix notation as

$$f = \{C\}\{X\} \qquad (2.13)$$

$$[A]\{X\} = \{P_0\} \qquad (2.14)$$

where $\{C\}$ is a row vector and $\{X\}$ and $\{P_0\}$ are column vectors, and

$$\{P_0\} = \overline{P}_0 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_{m+n} \end{bmatrix} \qquad (2.15)$$

Furthermore, because $[A]$ may be represented as a set of column vectors

$$[A] = \left[\overline{P}_1, \overline{P}_2, \ldots \overline{P}_j \ldots \overline{P}_{m+2n}\right] \qquad (2.16)$$

Equation (2.14) may be written

$$X_1\overline{P}_1 + X_2\overline{P}_2 \ldots + X_j\overline{P}_j \ldots + X_{m+2n}\overline{P}_{m+2n} = \overline{P}_0 \qquad (2.17)$$

where each X is a scalar.

Equations (2.11) and (2.12) for the example at hand may now be expressed as

$$X_1\overline{P}_1 + X_2\overline{P}_2 + X_3\overline{P}_3 + X_4\overline{P}_4 = \overline{P}_0 \qquad (2.18)$$

or

$$X_1\begin{bmatrix}1\\1\end{bmatrix} + X_2\begin{bmatrix}1\\2\end{bmatrix} + X_3\begin{bmatrix}-1\\0\end{bmatrix} + X_4\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}6\\8\end{bmatrix} \qquad (2.19)$$

Because the simplex procedure generates one basic feasible solution from another, the solution process must begin with a basic feasible solution. A basic feasible solution is characterized by having m+n variables with values greater than zero. In the vector representation, m+n linearly independent vectors will have multipliers greater than zero. These vectors are said to be in solution, or are said to be the basis of solution. The number of vectors in solution will always equal the number of linear restrictions. In the particular case when $d_j = 0$, $j = 1, 2, \ldots n$ in Equation (2.6) only m linearly independent vectors will be in solution.

The general tableau for the simplex procedure is shown at the top of the next page.

For any one cycle the tableau represents a particular basic feasible solution. The particular solution shown on the next page has as its basis vectors $\overline{P}_1$ through $\overline{P}_{m+n}$. The vectors making up a solution basis may be any m+n of the m+2n vectors. They need not, and generally will not, occur in sequence as shown below. A new basic feasible solution is obtained by the replacement of any vector already in solution by any vector that is not in solution.

$$\begin{array}{cccccccc}
 & & C_1 & C_2 \cdots & C_j \cdots & C_{m+2n} & & \\
 & \text{Basis} & \overline{P}_1 & \overline{P}_2 \cdots & \overline{P}_j & \overline{P}_{m+2n} & \overline{P}_0 & \theta \\
C_1 & \overline{P}_1 & X_{11} & X_{12} \cdots & X_{1j} \cdots & X_{1,m+2n} & X_{10} & \theta_1 \\
C_2 & \overline{P}_2 & X_{21} & X_{22} \cdots & X_{2j} \cdots & X_{2,m+2n} & X_{20} & \theta_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
C_i & \overline{P}_i & X_{i1} & X_{i2} \cdots & X_{ij} \cdots & X_{i,m+2n} & X_{i0} & \theta_i \\
\cdot & \cdot & & & & & & \\
\cdot & \cdot & & & & & & \\
\cdot & \cdot & & & & & & \\
C_{m+n} & \overline{P}_{m+n} & X_{m+n,1} & X_{m+n,2} \cdots & X_{m+n,j} \cdots & X_{m+n,m+2n} & X_{m+n,0} & \theta_{m+n} \\
(F_j - C_j) & & & & & & &
\end{array} \qquad (2.20)$$

where

C_j = coefficient of the jth variable of the objective function

$\overline{P}_j$ = column vector corresponding to the jth variable, taken from the original array of linear restrictions

X_{1j} through $X_{m+n,j}$ = scalar multipliers which, when multiplied by the corresponding vectors in solution, will yield the vector $\overline{P}_j$

C_i = coefficient of the variable in the objective function corresponding to the ith vector in solution

$\overline{P}_i$ = ith vector in solution

$\overline{P}_0$ = column vector defined by the column of constants on the right side of the original array of linear restrictions

$\theta_i = X_{i0}/X_{ij}$, where j is the index of the incoming vector. The column of θ's is the basis for the determination of the outgoing vector (2.21)

$$F_j = \left[C_1,\ C_2 \ldots C_i \ldots C_{m+n}\right] \begin{bmatrix} X_{1j} \\ X_{2j} \\ \cdot \\ X_{ij} \\ \cdot \\ X_{m+n,j} \end{bmatrix} \qquad (2.22)$$

$F_j - C_j$ = the criterion by which the incoming vector is determined. This is also the criterion used to identify the optimum solution

The operation of the simplex procedure will now be demonstrated by solving the example given in Equations (2.7), (2.8), and (2.9). For a first basic feasible solution let $X_2 = X_4 = 0$.* Equations (2.18) are solved next to determine the values of X_1 and X_3.

$$X_1\overline{P}_1 + 0\ \overline{P}_2 + X_3\ \overline{P}_3 + 0\ \overline{P}_4 = \overline{P}_0$$

or $$X_1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\2\end{bmatrix} + X_3\begin{bmatrix}-1\\0\end{bmatrix} + 0\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}6\\8\end{bmatrix}$$

yielding $X_1 = 8$ and $X_3 = 2$.

The basis for this solution is, then, $\overline{P}_1$ and $\overline{P}_3$, and

$$\overline{P}_0 = 8\overline{P}_1 + 2\overline{P}_3 = 8\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}6\\8\end{bmatrix}$$

or $X_{10} = 8$, $X_{30} = 2$

The next step is to calculate the scalar multipliers for the solution basis required to produce the vectors $\overline{P}_2$ and $\overline{P}_4$.

$$\overline{P}_2 = \begin{bmatrix}1\\2\end{bmatrix} = X_{12}\begin{bmatrix}1\\1\end{bmatrix} + X_{32}\begin{bmatrix}-1\\0\end{bmatrix}$$

from which $X_{12} = 2$ and $X_{32} = 1$

*A systematic method of determining an initial basic feasible solution is discussed in Section IV B.

$$\overline{P}_4 = \begin{bmatrix}0\\-1\end{bmatrix} = X_{14}\begin{bmatrix}1\\1\end{bmatrix} + X_{34}\begin{bmatrix}-1\\0\end{bmatrix}$$

from which $X_{14} = -1$ and $X_{34} = -1$

The first tableau, with symbolic scalar multipliers, is

		2	3	0	0	
	Basis	$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	$\overline{P}_0$
2	$\overline{P}_1$	X_{11}	X_{12}	X_{13}	X_{14}	X_{10}
0	$\overline{P}_3$	X_{31}	X_{32}	X_{33}	X_{34}	X_{30}

The first tableau, with numerical scalar multipliers, is

		2	3	0	0		
	Basis	$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	P_0	θ
2	$\overline{P}_1$	1	2	0	-1	8	4
0	$\overline{P}_3$	0	1	1	-1	2	2
(F_j-C_j)		0	1	0	-2	16	

Any vector $\overline{P}_1$, $\overline{P}_2$, $\overline{P}_3$, $\overline{P}_4$, $\overline{P}_0$ may now be expressed as a product of the first basic feasible solution basis $\overline{P}_1$ and $\overline{P}_3$ and the appropriate scalar multipliers. For example,

$$\overline{P}_4 = -1\overline{P}_1 - 1\overline{P}_3 = -1\begin{bmatrix}1\\1\end{bmatrix} - 1\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$$

Note that the scalar multipliers X_{10} and X_{30} under $\overline{P}_0$ are the values of the variables X_1 and X_3 for the first basic feasible solution. In the general case, then, the values of the variables corresponding to the vectors in solution are to be found in the column of scalar multipliers under the $\overline{P}_0$ vector for any basic feasible solution.

It should also be noted that the scalar multipliers in the columns under the vectors in solution will always form an identity matrix. This follows from the fact that in order to represent a vector in solution as a function of the vectors in solution the vector is equated to unity times itself.

The calculation of the (F_j-C_j)'s become

$$(F_1-C_1) = \begin{bmatrix}2,0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} - 2 = 0$$

$$(F_2-C_2) = \begin{bmatrix}2,0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} - 3 = 1$$

$$(F_3-C_3) = \begin{bmatrix}2,0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} - 0 = 0$$

$$(F_4-C_4) = \begin{bmatrix}2,0\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix} - 0 = -2$$

$$(F_0-C_0) = \begin{bmatrix}2,0\end{bmatrix}\begin{bmatrix}8\\2\end{bmatrix} - 0 = 16$$

where F_0-C_0 is the value of the objective function for the first basic feasible solution.

In order for the solution to be optimum, all F_j-C_j, with the exception of F_0-C_0, must be equal to or less than zero. But since F_2-C_2 is positive this initial solution is not optimum. It is now required to generate a new basic feasible solution. Because only two vectors may be in solution, one vector must go out of solution as a new vector comes in. Furthermore, a new solution will result in a decrease in the objective function only if $F_j-C_j > 0$ for the incoming vector. In general, the vector with the greatest value of F_j-C_j should be chosen as the vector coming into solution. In this example vector $\overline{P}_2$ is the only one which will improve the value of the objective function, so it comes into solution. To determine the vector going out of solution the θ_i's are calculated:

$$\theta_1 = X_{10}/X_{12} = 8/2 = 4$$

$$\theta_3 = X_{30}/X_{32} = 2/1 = 2$$

The vector in solution with the smallest positive value of θ must go out of solution. This is necessary because the presence of a negative θ would result in the incoming variable having a negative value, while the presence of a positive θ greater than the smallest positive θ would result in a negative value for one of the variables remaining in solution. In this example, then, $\overline{P}_2$ replaces $\overline{P}_3$. The intersection of the incoming and outgoing vectors defines the pivotal element, which is shown encircled.

To generate a new basic feasible solution a new identity matrix involving vectors $\overline{P}_2$ and $\overline{P}_1$ is required. In general, this matrix is constructed by first dividing the elements in the row containing the pivotal element, including the $\overline{P}_0$ element, by the pivotal element. This modified row is now placed in a position in the new tableau corresponding to its position in the old tableau. Next, multiples of this row are added to or subtracted from each of the remaining rows in the old tableau so as to produce zero elements in the column containing the pivotal element. As the new rows are generated, they are placed in the new tableau in positions corresponding to their positions in the old tableau. In this example, the first operation does not alter the row containing the pivotal element, since the pivotal element is unity. Next, adding minus one times the second row to the first row results in the new identity matrix involving $\overline{P}_2$ and $\overline{P}_1$, and yields the following new tableau:

		2	3	0	0	
	Basis	$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	$\overline{P}_0$
2	$\overline{P}_1$	1	0	-2	1	4
3	$\overline{P}_2$	0	1	1	-1	2
(F_j-C_j)		0	0	-1	-1	14

Since all F_j-C_j are equal to or less than zero this is the optimum solution. Therefore,

$$X_1 = 4,\ X_2 = 2,\ X_3 = X_4 = 0$$

and $$f = (F_0-C_0) = 14$$

which corresponds to point 0 in Figure 3.

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III. APPLICATION TO DESIGN PROBLEMS

A. GENERAL REMARKS

In order to apply the method of linear programming to the problem of determining the minimum weight of steel frames, a number of important relationships must be studied. In general, these relationships are not linear. On the other hand, the method of linear programming requires both the objective function and the side conditions to be linear. This contradiction will be discussed in Section III C.

B. ASSUMPTIONS AND LIMITATIONS

1. This study is limited to frames in one plane, composed of rigidly jointed members which are braced normal to their plane of action.

2. Only prismatic steel members whose cross sections have an axis of symmetry lying in the plane of the structure are considered.

3. All loads act in the plane of the structure.

4. Lateral loads act only at the joints.

5. Beams are assumed to be subjected only to bending (effects of axial force neglected).

6. Further restrictions applying to particular problems will be discussed as they arise.

C. THE OBJECTIVE FUNCTION

The objective function which is to be minimized in order to determine a minimum-weight steel frame is that function which expresses the total weight of the frame:

$$F_w = \sum_{k=1}^{n_b} C_k D_k + \sum_{i=1}^{n_c} C_i D_i \qquad (3.1)$$

where F_w = weight of frame
C_k = length of k^{th} beam in feet
D_k = weight per foot of k^{th} beam
n_b = number of beams in frame
C_i = length of ith column in feet
D_i = weight per foot of ith column
n_c = number of columns in frame

Although Equation (3.1) is exact, it cannot be used in its present form because the side conditions provided by the mechanism equations relate the moment capacities of the individual members and not their weights. In order to effect compatibility between the objective function and the side conditions the weights of the members must be expressed as functions of their moment capacities. These relationships will be considered separately for beams and columns.

In Figure 5 the weight per foot, D, of wide-flange and other I-shapes, as given in Reference 18, are shown plotted against their plastic moment capacities M_p. The solid curve is drawn as a best fit of the "economy" sections. The economy section is the lightest standard shape which furnishes a plastic resisting moment which equals (or exceeds) the required value. In the investigation of design for minimum weight, only the economy sections need be considered for beams. The curve of best fit for these sections has been determined in Reference 20 to be

$$D = 1.2\, M_p^{2/3} \qquad (3.2)$$

Since this equation is nonlinear it cannot be substituted into the objective function. Fortunately, the range in M_p from the smallest shape that could be used to the largest shape which probably would be used for a particular member under a particular loading is limited. For uniform load this range would normally be from $WL^2/16$ to $WL^2/8$, although in the case of a short span sandwiched between two longer spans the upper limit could be greater. A number of trials showed that a straight line gives a good fit to the plot of economy sections for particular conditions of geometry and loading and for a reasonable range of M_p. Line AB of Figure 6 is typical. Therefore, we may write

$$D_k = a_k + b_k M_{pk} \qquad (3.3)$$

where M_{pk} = plastic moment capacity of the k^{th} beam.

Further complications arise in the case of a similar representation for beam-columns. Not only is the relationship between weight per foot and moment capacity nonlinear, but further complications of axial loading and possible side-sway of the structure are added. As has been pointed out, changes in size of members in proceeding from one basic feasible design to another affect the distribution of moments over the structure. To a lesser extent, the distribution of axial loads is also affected. However, it will be assumed here, as is usually done, that axial loads remain constant with change in member sizes.

In the minimum-weight problem the moment capacities of the individual members are the unknowns which are sought. Defining the moment capacity of a beam-column as that moment M_0 which can be carried by the section in conjunction with the axial load P, a suitable relationship may be determined. In using such a relationship, it must be understood that the minimum-weight solution then yields the value of M_0 required in conjunction with the given axial load for a particular beam-column. On the other hand, the minimum-weight solution yields for beams required values of the fully plastic moment M_p.

It now becomes necessary to develop weight-moment equations for a given beam-column of I shape. This column may be subject to a wide range in loading and end support conditions.

Massonet[17] has proposed for plastic design an adaptation of the interaction formula for elastic behavior of the beam-column. In order to use his equation the magnitude and sense of the moment at each end of the beam-column must be known. Although these could be determined at any stage of the simplex procedure, they would certainly vary from one basic feasible solution to the next. This would require that a new objective (weight) function be established for each new basic feasible solution. While this could be done, it would significantly increase the time required to obtain an optimum design. It is the author's opinion that, should Massonet's interaction formula be used, the slight gain in precision over the method discussed next does not warrant the resulting increase in solution time and complexity.

Galambos and Ketter[9] developed interaction formulas relating moment capacity M_0 and axial compression P for the following three specific cases:

Case I. Columns bent in double curvature, for which $M = M_0$ at each end,

Case II. Columns for which $M = M_0$ at one end and $M = 0$ at the other, and

Case III. Columns bent in single curvature, for which $M = M_0$ at each end.

Case I is the most favorable in respect to the value of M_0 that can be developed in conjunction with a given value of P. Case II follows, while Case III columns develop the smallest M_0 for a given value of P. Therefore, the difficulty mentioned above can be met by bracketing beam columns according to the Galambos-Ketter equations. Thus, a beam-column bent in double curvature for which $M = M_0$ at one end, while $0 < M < M_0$ at the other, will be classified as Case II, which is the nearer less favorable case. Similarly, a beam-column bent in single curvature for which $M = M_0$ at one end, while $0 < M < M_0$ at the other, will be classified as Case III, the nearer less favorable case. Although this would appear to result in some loss in precision, it can be justified on two counts. First, Massonet's equation is approximate (since it is a simple extension of the interaction formula for elastic behavior), while the Galambos-Ketter equations are "exact" (for the particular shape, 8 WF 31, and for the particular distribution of cooling residual stress which was assumed). Secondly, the specification of the American Institute of Steel Construction, which is the most widely accepted code for plastic design of steel frames, incorporates the Galambos-Ketter formulas.

In Figure 7 the weights per foot D of wide flange and other I shapes are shown plotted against the allowable moment capacity M_0 for a particular load P and a particular beam-column of height L. The points represented in this figure were plotted in accordance with the equation below for Case II columns:

$$M_{0(\max.)} = M_p \left[B - G(P/P_y)\right] \qquad (3.4)$$

where $$B = 1.13 + \frac{L/r_x}{3{,}080} + \frac{(L/r_x)^2}{185{,}000} \qquad (3.5)$$

$$G = 1.11 + \frac{L/r_x}{190} - \frac{(L/r_x)^2}{9{,}000} + \frac{(L/r_x)^3}{720{,}000} \qquad (3.6)$$

P_y = product of the cross-sectional area and the yield stress of the steel

r_x = radius of gyration with respect to the x-x axis

A similar plot may be made for Case I and Case III columns. However, the majority of columns in building frames exhibit end conditions which classify them as Case II columns. In other words, nearly all such columns are bent in double curvature, with $M = M_0$ at one end and $0 < M < M_0$ at the other. Case II is on the safe side in all such cases, as has been explained previously. For the rare case in which a minimum-weight design contains a column or columns which exhibit single curvature, with $M = M_0$ at one end and $0 < M < M_0$ at the other, the designer must check the adequacy of such columns against the Case III formula. This is because the Case II formula is unsafe for columns in single curvature except when $M = 0$ at one end.

As in the case of beams, only the economy sections need be considered for columns for the minimum-weight problem. This is because the economy sections always equal or exceed the non-economy sections of equal weight with respect to the section properties r_x, M_p, and P_y. Because the frame is considered braced normal to its plane of action, the radius of gyration for the weak axis, r_y, need not be considered. These section properties, together with the given axial load, determine the allowable moment capacity M_0 (Equation 3.4). The curve of best fit is again nonlinear. But, as for beams, the range of M_0 required for a particular member is limited to the extent that a straight line provides a reasonably good fit over this range. The range for the column will normally extend from $M_0 = 0$ to the maximum M_0 the column would receive from adjacent beams. Line C-D, Figure 7, is a typical best-fit straight line. The equation is

$$D_i = a_i + b_i M_{0i} \tag{3.7}$$

where M_{0i} = that moment capacity which can be carried by the i^{th} column in conjunction with its axial load.

Failure of a frame may result from overall instability involving sidesway at an ultimate load less than that which would be carried if the frame were braced to prevent sidesway. At the present time the ultimate load with respect to this form of instability cannot be predicted precisely. However, in Reference (5) the following expression is suggested for columns subject to sidesway:

$$2P/P_y + L/70r_x \leq 1 \tag{3.8}$$

This relation is conservative for frames of proportions likely to be found in practice. It has been adopted by the AISC Specification Committee as an interim provision.

For a limitation of the above type the relationship between D_i and M_0 must be modified slightly. Essentially this limitation may require that the lower limit for M_0 be greater than zero. The lower limit is calculated by solving Equation (3.8) for P_y, selecting the minimum-weight section to provide the required P_y, and then using Equation (3.4) to determine M_0. A different straight line segment is required to fit the more restricted range of M_0. Line E-F in Figure 7 represents this new relationship. The more general equation is written

$$D_i' = a_i' + b_i' M_{0i} \tag{3.9}$$

where, for the special case when $M_{0(min.)}$ is zero, $D_i' = D_i$. The objective function, Equation (3.1), is now expressed as

$$F_w = \sum_{k=1}^{n_b} (C_k a_k + C_k b_k M_{pk}) + \sum_{i=1}^{n_c} (C_i a_i' + C_i b_i' M_{0i}) \tag{3.10}$$

when substitutions are made for D_k and D_i'.

Because all terms $C_k a_k$ and $C_i a'_i$ are constants for a particular problem, these terms may be dropped in determining the values of M_{pk} and M_{0i} which yield minimum frame weight. In other words, the value of F_w is not sought, but rather the values of M_{pk} and M_{0i} which yield the minimum F_w. Therefore, the objective function may be written

$$F_w' = \sum_{k=1}^{n_b} C_k b_k M_{pk} + \sum_{i=1}^{n_c} C_i b_i' M_{0i} \tag{3.11}$$

where F_w' = frame weight minus a constant. In order to more conveniently express the objective function in the linear programming tableau, Equation (3.11) will be written

$$F_w' = \sum_{j=1}^{n} C_j X_j \tag{3.12}$$

where $C_j = C_k b_k$ if j^{th} member is a beam

$C_j = C_i b_i'$ if j^{th} member is a column

$X_j = M_{pk}$ if j^{th} member is a beam

$X_j = M_{0i}$ if j^{th} member is a column

$n = n_b + n_c$

D. LINEAR RESTRICTIONS

The linear restrictions or side conditions originate from the mechanism inequalities (Section II A), from sidesway requirements, and, possibly, from other design requirements. It was shown in Section III C that sidesway of the frame requires a greater-than-zero lower limit for the M_0 capacity of a column. This restriction must, therefore, be imposed in the form of an additional inequality for each column in the frame subject to sidesway, and added to the mechanism inequalities. Further restrictions could arise in the form of arbitrary limitations set by the designer. For example, it may be desirable to limit either the maximum or minimum moment capacity, or both, of one or more members, in which case additional inequalities are required. However, since these inequalities do not originate from the physical problem they should be checked against the mechanism and sidesway inequalities for inconsistencies.

To insure compliance with the criteria of yield and equilibrium (Section II A), inequalities representing all possible modes of failure should be included in the formulation of the problem. If, however, an attempt is made to provide rigorously for this requirement in a frame with, say, ten or more members, a very large system of equations (of the order of 10^4) will result. Not only will the time required to generate these equations become excessive, but the storage capacity of even a large computer will be exceeded.

The possibilities of making admissible reductions in the number of inequalities will now be considered. The solution of interest is a frame of n structural variables involving (m+n) hyperplanes. For illustration, however, a two-dimensional space will be considered (Figure 4). The line segments AB, BC, CD, and DE form the lower bound to the solution space. Line FG, the objective function, is shown positioned at the minimal solution. The bases for these minimal solutions are the equations represented by lines CD and DE. If either of these two equations is omitted from the problem formulation, the correct minimal solution will not be obtained. For example, if the equation represented by line CD is inadvertently omitted, the minimal solution is erroneously placed at point H, as shown by line F'G'. But suppose that instead of the equations represented by CD or DE, some other equation, or equations, is omitted. For example, consider the equation represented by line AB to be omitted. If the first basic feasible solution is initiated on the vertical axis, a different route to the minimal solution is followed, namely ICD, but the solution is still correctly placed at D.

It is obvious from the discussion above that no error arises from the omission of those equations which are not a part of the basis of solution. Although the omission of an equation which is a part of the basis of solution results in an erroneous (unsafe) design, it should be noted that this solution results in a structure whose weight is less than the true minimum. This conclusion follows from the convex shape of the solution space. Therefore, it may be concluded that the optimal solution obtained with a partial set of equations is a lower bound on the minimum-weight frame. Furthermore, if a statically admissible moment diagram for the entire frame can be found for which the moment nowhere exceeds the values of the plastic moments prescribed for each member, the minimum-weight solution is also an upper bound, and is therefore the optimal solution. This suggests that if a designer can, by experience, select the dominant equations and obtain a minimal solution, he can check compliance with the yield criterion by the statical method or by moment balancing techniques.[1] Should the yield criterion be satisfied, he has indeed found the minimum-weight solution without resorting to a large system of equations. If the yield criterion is not satisfied, he has two choices. He might try to include the dominant mode or modes of collapse initially omitted. He would be materially assisted in this effort by a study of the positions in the frame where the moment capacities are exceeded. On the other hand, he might elect simply to increase the moment capacities of the members whose capacities are exceeded. The weight of this frame would then form an upper bound to the minimum-weight solution, and if the upper and lower bounds are deemed to be sufficiently close the upper bound can be accepted as the final solution.

Finally, the system of equations might be reduced by solving the problem in parts. It is well known that the flexure induced by a particular load diminishes rapidly in members of a frame at some distance from the point of loading. Therefore, as is usually done in elastic design, a tall frame might be solved story by story. This idea has been suggested by Boulton.(2)

Although it may often be advisable to consider reducing the number of linear restrictions, it may be possible in some situations to include all of them. It should be recalled that consideration of all possible modes of collapse eliminates the operation of checking the solution against the yield criterion. Therefore, a systematic generation of the mechanism inequalities which will insure the inclusion of all possible modes of collapse is desirable. The basic modes of collapse of a frame by beam, panel, and joint mechanism are shown in Figures 8, 9, and 10. All other possible modes of collapse may be generated by combinations of one or more of these basic mechanisms.

The mechanism inequalities were represented in a two-dimensional array in Section II C for the purpose of linear programming, the first subscript denoting the equation number and the second the member number. This notation allows the summation of the rotations of all the hinges in one member to be represented by a single coefficient, and is adequate for a simplex-procedure solution process. However, in order to formulate a basis for generating equations from the basic equations, the individual plastic hinge positions must be designated. This specification is necessary because the various mechanisms are identified by the positions of their hinges along with the magnitudes of the hinge rotations. Therefore, a third subscript, k, is added to the above notation to denote the hinge position. The A matrix of the linear restrictions,

$$\left[a_{ij}\right] \left\{X_j\right\} \geq \left\{b_i\right\} \qquad \begin{array}{l} i = 1,2,\ldots.m \\ j = 1,2,\ldots.n \end{array} \qquad (3.13)$$

is expanded by letting

$$a_{ij} = \sum_{k=1}^{r+1} a_{ijk} \qquad (3.14)$$

The symbol r in Equation (3.14) denotes the number of joints in the frame. The summation is taken to $r+1$ in order to include the intermediate beam-hinge. The resulting equations with expanded A matrix are shown atop the next page.

All elements a_{ij} of the A matrix of the mechanism inequalities are positive because, regardless of the direction of rotation of the corresponding hinge, positive work results. But for the purpose of equation generation the direction of rotation must be identified, since the algebraic summation of corresponding elements a_{ijk} of different equations must allow for the increase, decrease, or elimination of rotation at a hinge position. (The hinge intermediate between joints is an exception. Because such a hinge is common to only one basic equation, the corresponding matrix element will always be positive regardless of the direction of rotation.) Hinges at a joint are considered positive if they result from clockwise rotation of the joint.

To assure generation of the equations representing all modes of failure, all possible combinations of the basic equations will be made, after which redundant combinations

$$\begin{bmatrix} \left(|a_{111}|+|a_{112}|\cdot\cdot+|a_{11r+1}|\right)\left(|a_{121}|+|a_{122}|\cdot\cdot+|a_{12r+1}|\right)\cdots\left(|a_{1n1}|+|a_{1n2}|\cdot\cdot+|a_{1nr+1}|\right) \\ \left(|a_{211}|+ a_{212}|\cdot\cdot+|a_{21r+1}|\right)\left(|a_{221}|+|a_{222}|\cdot\cdot+|a_{22r+1}|\right)\cdots\left(|a_{2n1}|+|a_{2n2}|\cdot\cdot+|a_{2nr+1}|\right) \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \left(|a_{m11}|+ a_{m12}|\cdot\cdot+|a_{m1r+1}|\right)\left(|a_{m21}|+|a_{m22}|\cdot\cdot+|a_{m2r+1}|\right)\cdots\left(|a_{mn1}|+|a_{mn2}|\cdot\cdot+|a_{mnr+1}|\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{3.15}$$

will be eliminated.

A beam equation may enter a particular combination only with a positive sign. On the other hand, a panel equation or a joint equation may enter a particular combination with either a positive or a negative sign. This difference occurs because beams are considered to be subjected only to gravity loads, while lateral loads may act from either left or right, and joint rotations may be either clockwise or counterclockwise.

Accordingly, in determining the number of combinations of the basic equations, we have two possibilities for each beam mechanism (enter, not enter) and three for each panel mechanism and each joint mechanism (enter positively, enter negatively, not enter). Therefore, the number of basic equations and all their combinations is

$$(2^p \cdot 3^{q+r} - 1) \tag{3.16}$$

where p = number of beam equations

q = number of panel equations

r = number of joint equations

Each resulting combination is subjected to the following criteria for elimination:

(1) Any equation which has as many hinges at a joint as there are members connected by the joint is eliminated. While these equations would be admissible in the simplex procedure, they are redundant because the procedure guarantees that all variables will be greater than or equal to zero.

(2) If a beam mechanism forms in which the required three hinges occur in the beam itself, or in which the hinge at either end (or both) is superseded by hinges at that joint in all members framing into it, and as many as one other hinge exists in the structure, the equation is eliminated. Since such an equation represents a system with two or more degrees of freedom, it is redundant. This redundancy results because failure is already considered with respect to all single-degree-of-freedom systems. As an example, the equations representing the modes of failure shown in Figures 11 and 12 would be eliminated.

(3) In a multistory building, if any two or more panel mechanisms do not have at least one column continuous without hinges across the joint or joints separating them, then the equation is eliminated. This equation also represents a system with two or more degrees of freedom. As an example, the equation representing the mode of failure shown in Figure 13 would be eliminated.

(4) When the A matrix is transformed from the a_{ijk} form to the a_{ij} form, each equation is checked against all previously generated equations for proportionality of terms in the left member, i.e., the terms representing the internal work during the virtual displacement. If these terms are found to be proportional to the corresponding terms of any

previously generated equation, the more restrictive equation is retained and the other is eliminated.

It should be noted that this procedure is intended primarily for machine computation.

A simple example will be considered (Figure 14). The formulation of the equation for the beam mechanism is

$$|3\beta X_1| + |-1\beta_1| + |2\beta X_1| \geq \frac{2}{3} L\beta W_1 = \beta b_1$$

This equation is, of course, equivalent to $6X_1 \geq b_1$. However, for the purpose of generating equations by combining elementary mechanisms, the energy terms corresponding to the various hinge rotations must remain separate. The remaining basic equations are formulated similarly. The symbolic representation of the nonzero elements of the basic equations are shown in the expanded A matrix:

$$\begin{bmatrix} (|a_{111}| + |a_{112}| + |a_{113}|) & & & \\ (|a_{212}|) & (|a_{222}|) & & \\ (|a_{313}|) & & (|a_{333}|) & \\ & (|a_{422}|) & (|a_{433}| + |a_{434}|) & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \begin{matrix} \text{(A)} \\ \text{(B)} \\ \text{(C)} \\ \text{(D)} \end{matrix}$$

The numerical values of the elements of the A matrix are

$$\begin{bmatrix} 3 & -1 & 2 & & & \\ & 1 & & 1 & & \\ & & 1 & & 1 & \\ & & & -1 & -0.5 & -0.5 \end{bmatrix}$$

The equations in this matrix correspond, respectively, to the following elementary mechanisms:

Equation (A) beam mechanism
Equation (B) joint 2 mechanism
Equation (C) joint 3 mechanism
Equation (D) panel mechanism

For the problem at hand, only sidesway right is considered. Therefore, the panel-mechanism equation will act negatively in combination. Thus the number of elementary mechanisms and their combinations is

$$(2^{p+q} \cdot 3^r - 1) = (2^{1+1} \cdot 3^2 - 1) = 35$$

Only a representative number of these mechanisms will be considered. In considering the elimination of a particular combined mechanism by one or more of the first three criteria, only the left side of the equation need be studied.

Beam and panel combination: Equation (A) + Equation (D)

$$(3 + |-1| + 2 + 0)X_1 + (0 + |-1| + 0 + 0)X_2 + (0 + 0 + |-.5| + |-.5|)X_3$$

which is eliminated by either criterion (1) or criterion (2).

Beam mechanism combination: Equation (A)

$$(3 + |-1| + 2 + 0)X_1$$

This satisfies the first three criteria, and is transferred to the a_{ij} equation array as

$$6X_1 \geq b_1 \qquad \text{(E)}$$

Joint 2 and panel combination: Equation (B) + Equation (D)

$$(0+1+0+0)X_1 + (0+0+|-.5|+|-.5|)X_3$$

This satisfies the first three criteria, and is transferred to a_{ij} equation array as

$$X_1 + X_3 \geq b_4 \qquad \text{(F)}$$

Combination of beam, panel, and both joints: Equation (A) + Equation (B) - 2 Equation (C) + Equation (D)

$$(3+0+0+0)X_1 + (0+0+|-2.5|+|-.5|)X_3$$

This satisfies the first three criteria, and is transferred to a_{ij} equation array as

$$3X_1 + 3X_3 \geq b_1 + b_4 = 4b_4$$

$$X_1 + X_3 \geq 4b_4/3 \qquad \text{(G)}$$

Examination of the three combinations represented by Equations (E), (F), and (G), according to criterion (4), results in the elimination of Equation (F).

After all thirty-five combinations have been made, the following array survives the criteria for elimination:

$$\begin{bmatrix} 6 & & \\ 3 & 1 & 2 \\ 4 & & 2 \\ 5 & 1 & \\ & 1 & 1 \\ 5.5 & & 0.5 \\ 3 & & 3 \\ 0.5 & 1 & 0.5 \\ 1.5 & & 0.5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \geq \begin{bmatrix} b_1 \\ b_1 \\ b_1 \\ b_1 \\ b_4 \\ b_1+b_4 \\ b_1+b_4 \\ b_4 \\ b_4 \end{bmatrix}$$

A program which has been prepared for the 7090 computer takes the equations corresponding to the basic mechanisms as input and makes all possible combinations of them, rejecting those which are revealed to be redundant in accordance with the criteria established above.

E. ADJUSTMENT OF INITIAL SOLUTION TO ACTUAL SECTIONS

Although the objective function as presently formulated is related to the properties of the standard sections, it is still based upon a continuity of these properties. Generally, the minimum-weight solution for a particular structure will place the required moments of the various members intermediate between the moment capacities of standard sections. It should also be recalled that the weight-moment capacity relationships are best-fit straight lines.

For this discussion, the initial solution will be defined as the solution composed of the lightest members all of whose moment capacities equal or exceed the theoretical moments obtained from the linear-programming solution. This initial solution is usually not the minimum-weight solution. Suppose these next-largest moment-capacity sections are selected for an arbitrary number of members in a multi-member structure. Then it is quite possible that, for the remainder of the members, the moment capacities of economy sections lighter than those adequate to satisfy the original array of linear restrictions may suffice. This possibility becomes more evident if a particular example is investigated. Consider a structure composed of two members for which the minimum-weight solution places the theoretical moments slightly above the available moment capacities of the required standard sections. Suppose further that one member is in a region of sections with large intervals between moment capacities and section weights, while the other is in a region of sections with smaller moment capacities and smaller intervals between moment capacities and section weights. This is not an uncommon condition. The solution consisting of the sections with moment capacities next below the theoretical moments must be ruled out, since the linear programming solution gives a moment distribution which just satisfies the given linear restrictions. However, if the next larger moment capacity section is selected for the larger of the two members, it is possible that the resulting reduced moment required for the smaller member can be furnished by a section which may be lighter than the corresponding section in the initial solution.

The most important consideration in determining the minimum-weight solution, once the initial solution is in hand, is the bandwidth of sections which must be surveyed. Here the bandwidth is defined as the number of consecutive standard sections which are to be considered for a given member. The bandwidth may be constant for all members or it may vary. As the number of members in the frame increases, the bandwidth becomes more critical with respect to the feasibility of obtaining the minimum-weight solution. For a constant bandwidth of four members for a three-member frame, only 64 combinations need be considered. For a constant bandwidth of four members for a ten member frame, however, over one million combinations would be possible.

A definitive method of determining the bandwidth which will assure the minimum-weight solution has not been found. However, experience which may provide a guide has been obtained in this investigation. The selection of the bandwidth which will probably include the minimum-weight solution is governed by the following considerations:

(1) The position of the theoretical moments with respect to the adjacent moment capacities of standard sections, and

(2) The position of the theoretical moments with respect to the more efficient sections within the range of sections considered. As would be expected, the closer the theoretical moments are to the next larger available moment capacities, the narrower may be the bandwidth which probably includes the minimum-weight solution.

Sections represented by points below the best-fit lines in Figures 6 and 7 are the more efficient sections. These have moment capacities greater than those predicted by the best-fit straight lines. The closer the theoretical moments are to the moment capacities of these more efficient sections, the narrower may be the bandwidth which probably includes the minimum-weight solution. Bandwidths are discussed further in Section V A.

It is proposed here to use the linear programming method formulated in Sections III B, III C, and III D to determine an initial solution to the minimum-weight problem. This initial solution will be in the immediate vicinity of the minimum-weight solution. A suitable bandwidth is then chosen for each member in the structure. Although these bandwidths define a finite number of solutions, only those which satisfy the given linear restrictions are feasible solutions. These feasible solutions are then compared to obtain the minimum-weight solution.

Except for structures composed of very few members, the electronic computer must be used in obtaining the solution outlined above.

F. DISTRIBUTION OF LOADS

As has been mentioned in Section III B, lateral loads are applied only at the joints. On the other hand, the distribution of loading on the beams is arbitrary. Therefore, it is incumbent on the designer to so locate the intermediate plastic hinges that the moment capacities of the resulting minimum-weight

SUPPORTING DATA

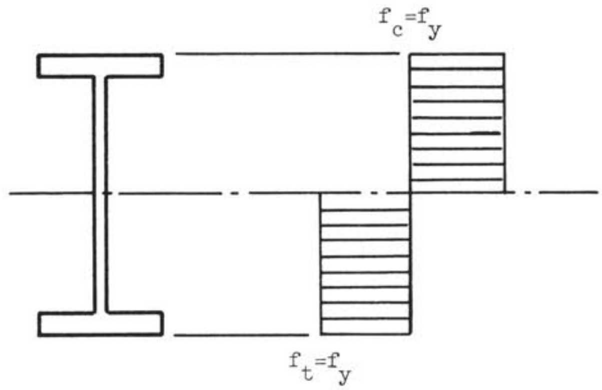


FIGURE 1. ASSUMED STRESS DISTRIBUTION

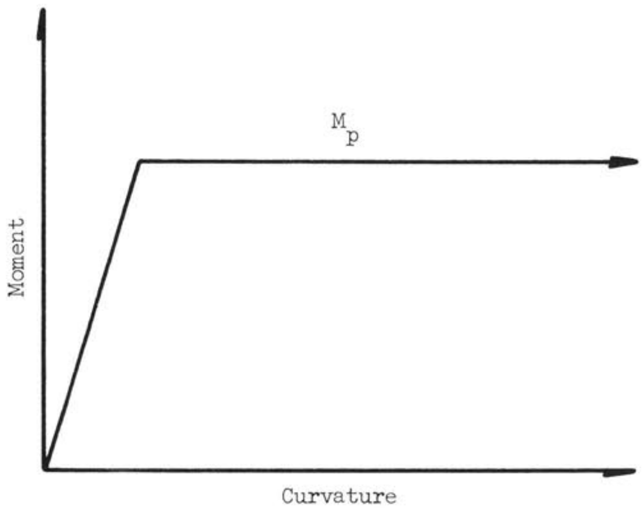


FIGURE 2. IDEALIZED MOMENT CURVATURE DIAGRAM

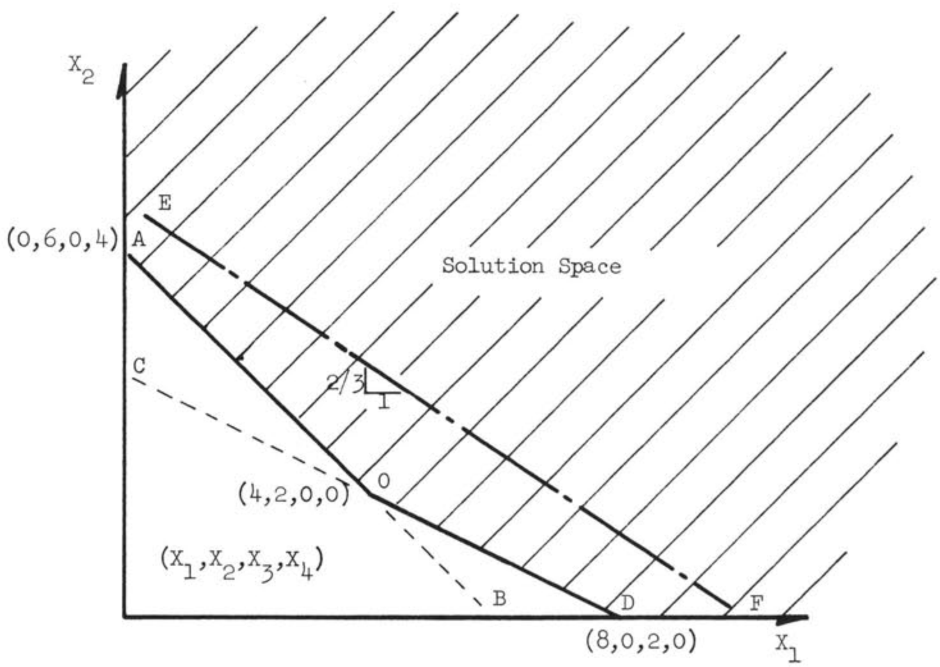


FIGURE 3. NUMERICAL PROBLEM FOR SIMPLEX METHOD

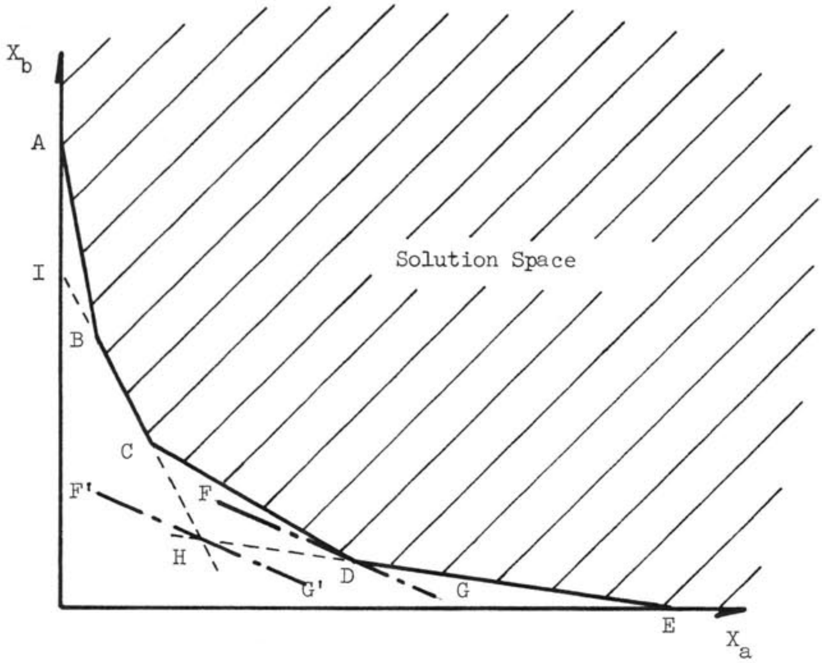


FIGURE 4. STUDY OF INCOMPLETE SYSTEM OF LINEAR RESTRICTIONS

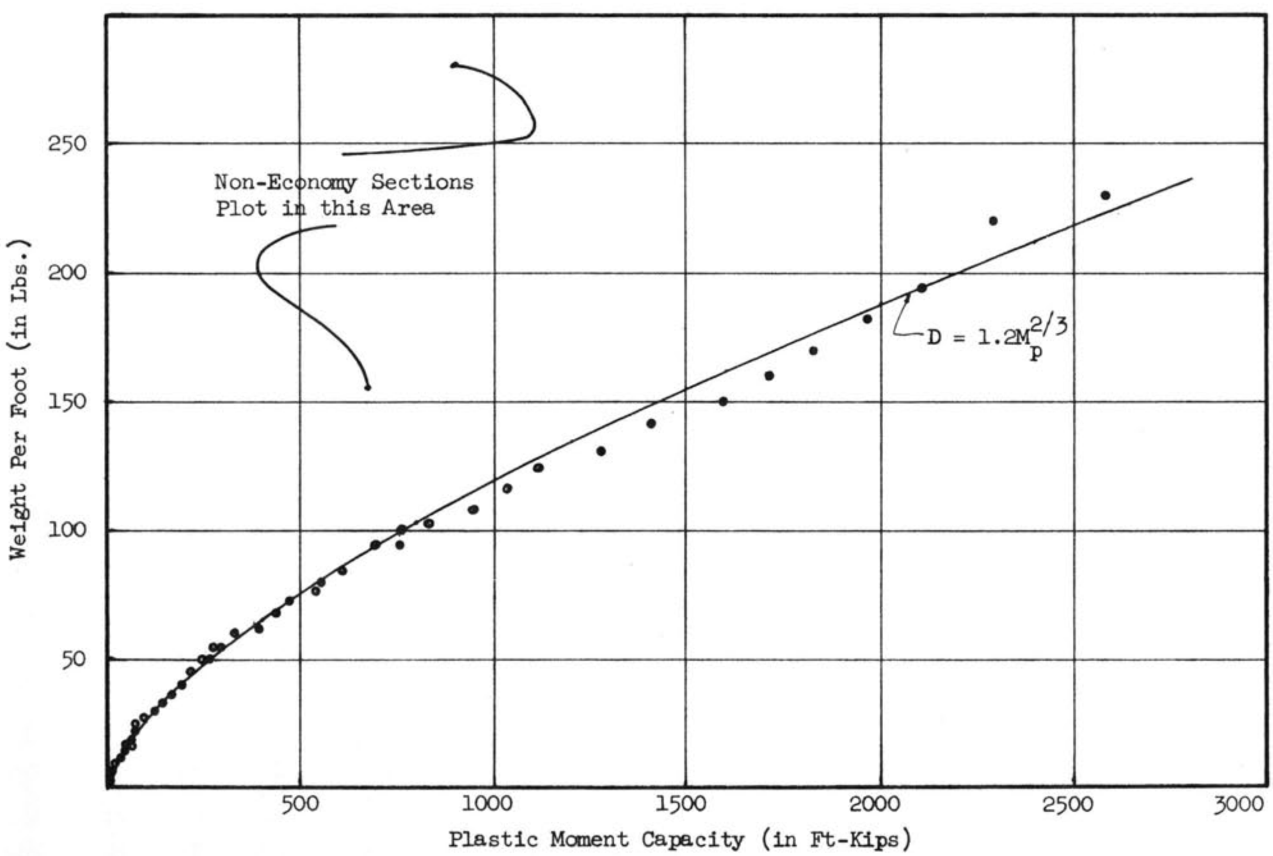


FIGURE 5. WEIGHT PER FOOT VS PLASTIC MOMENT CAPACITY OF ECONOMY SECTIONS

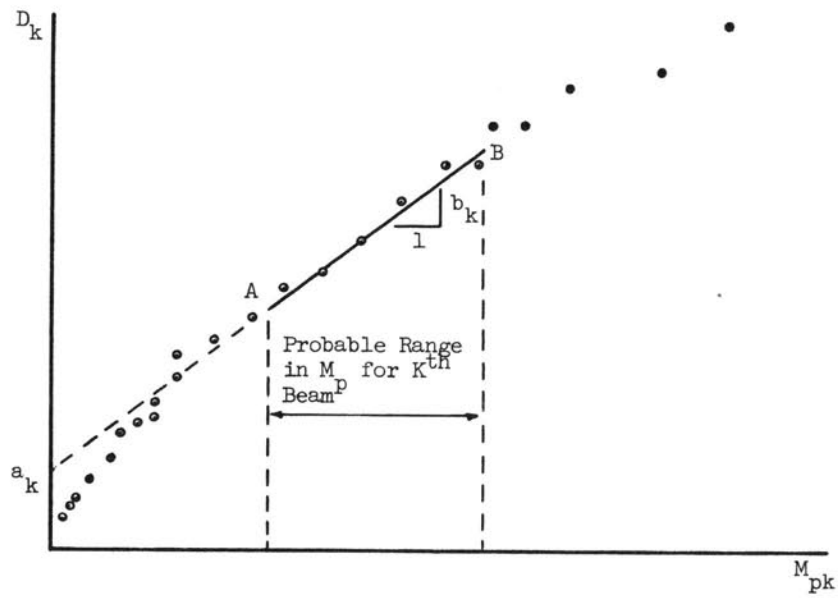


FIGURE 6. RELATIONSHIP BETWEEN D_k AND M_{pk} FOR BEAMS

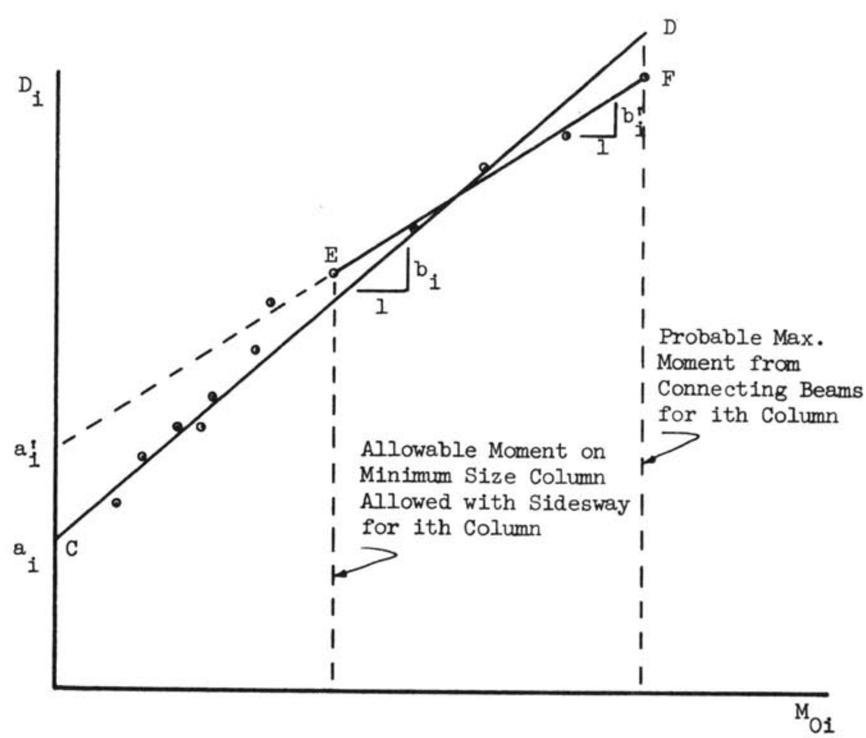


FIGURE 7. RELATIONSHIP BETWEEN D_i AND M_{0i} FOR BEAM COLUMNS

FIGURE 8. BEAM MECHANISM

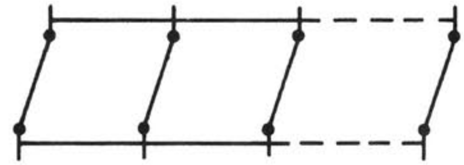


FIGURE 9. PANEL MECHANISM

FIGURE 10.

JOINT MECHANISMS

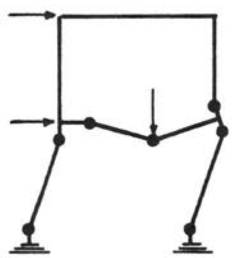


FIGURE 11.

FIGURE 12.

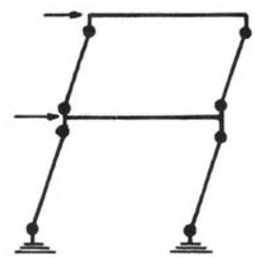
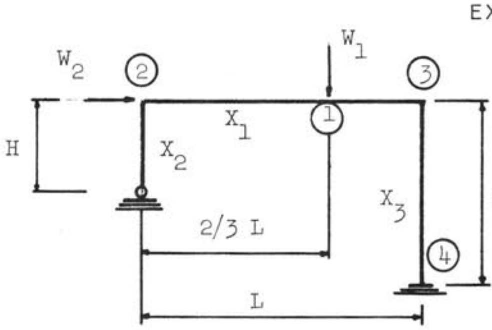


FIGURE 13.

EXAMPLES OF TWO-DEGREE-OF-FREEDOM SYSTEMS



$b_1 = 2/3\ LW_1$

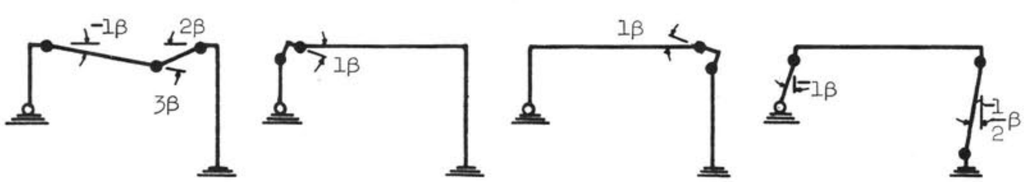
$b_2 = b_3 = 0$

$b_4 = HW_2$

$b_1 = 3b_4$

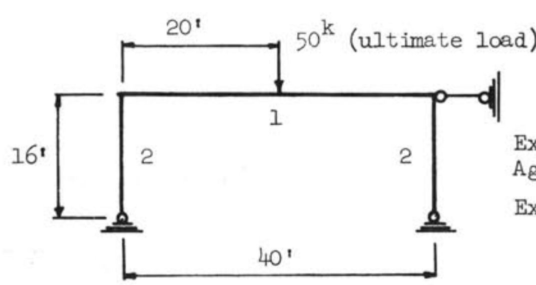
(Encircled numbers represent hinge positions.)

FIGURE 14. GIVEN CONDITIONS FOR EXAMPLE OF EQUATION GENERATION



EQ. (A) EQ. (B) EQ. (C) EQ. (D)

FIGURE 15. MODES OF FAILURE FOR BASIC EQUATIONS



Example 1 - Frame Braced Against Sidesway

Example 2 - Frame Unbraced

FIGURE 16. GIVEN CONDITIONS FOR EXAMPLES 1 AND 2.

		4	5	0	0	0	100	100	100		
		$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$	$\bar{P}_4$	$\bar{P}_5$	$\bar{P}_6$	$\bar{P}_7$	$\bar{P}_8$	$\bar{P}_0$	θ
←100	$\bar{P}_6$	4	0	-1	0	0	1	0	0	1000	250
100	$\bar{P}_7$	2	2	0	-1	0	0	1	0	1000	500
100	$\bar{P}_8$	0	1	0	0	-1	0	0	1	54	∞
(F_j-C_j)		594 ↑	295	-100	-100	-100	0	0	0		
4	$\bar{P}_1$	1	0	-1/4	0	0	1/4	0	0	250	∞
100	$\bar{P}_7$	0	2	1/2	-1	0	-1/2	1	0	500	250
←100	$\bar{P}_8$	0	1	0	0	-1	0	0	1	54	54
(F_j-C_j)		0	295 ↑	49	-100	-100	-149	0	0		
4	$\bar{P}_1$	1	0	-1/4	0	0	1/4	0	0	250	∞
←100	$\bar{P}_7$	0	0	1/2	-1	2	-1/2	1	-2	392	196
5	$\bar{P}_2$	0	1	0	0	-1	0	0	1	54	-54
(F_j-C_j)		0	0	49	-100	195 ↑	-149	0	-295		
4	$\bar{P}_1$	1	0	-1/4	0	0	1/4	0	0	250	-1000
←0	$\bar{P}_5$	0	0	1/4	-1/2	1	-1/4	1/2	-1	196	784
5	$\bar{P}_2$	0	1	1/4	-1/2	0	-1/4	1/2	0	250	1000
(F_j-C_j)		0	0	0.25 ↑	-2.5	0	-100.25	-97.5	-100		
4	$\bar{P}_1$	1	0	0	-1/2	1	0	1/2	-1	446	
0	$\bar{P}_3$	0	0	1	-2	4	-1	2	-4	784	
5	$\bar{P}_2$	0	1	0	0	-1	0	0	1	54	
(F_j-C_j)		0	0	0	-2	-1	-100	-98	-99	Optimum	

FIGURE 17. SOLUTION FOR EXAMPLE 2

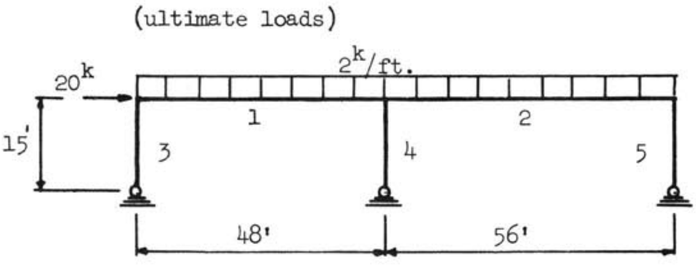


FIGURE 18. GIVEN CONDITIONS FOR EXAMPLE 3

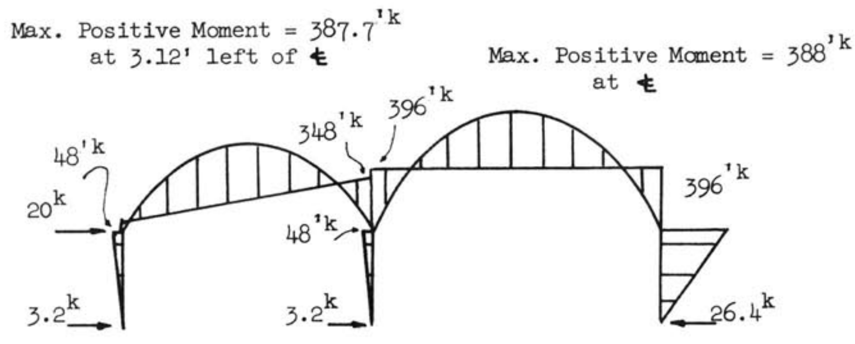
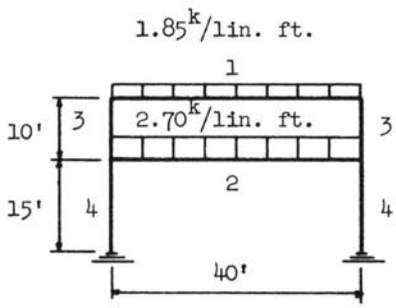


FIGURE 19. ADMISSIBLE MOMENT DIAGRAM AND COLUMN SHEARS FOR EXAMPLE 3

$$
\begin{array}{c}
\text{No.} \\ (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \\ (7) \\ (8) \\ (9) \\ (10) \\ (11) \\ (12) \\ (13) \\ (14) \\ (15) \\ (16) \\ (17) \\ (18) \\ (19) \\ (20) \\ (21) \\ (22) \\ (23) \\ (24) \\ (25) \\ (26) \\ (27)
\end{array}
\begin{bmatrix}
 & & 1 & 1 & 1 \\
1 & & & 1 & 1 \\
1 & 1 & 1 & & 1 \\
 & 1 & 1 & 1 & \\
1 & 2 & 1 & & \\
4 & & & & \\
3 & & 1 & & \\
2 & 1 & 1 & 1 & \\
2 & 2 & & 2 & \\
3 & 1 & & 1 & \\
3 & & & 1 & 1 \\
4 & 1 & & & 1 \\
2 & 1 & & 2 & 1 \\
4 & 2 & & & \\
 & 4 & & & \\
1 & 3 & & 1 & \\
 & 3 & & & 1 \\
1 & 2 & & 1 & 1 \\
1 & 4 & 1 & & \\
2 & 4 & & & \\
1 & 2 & 1 & & 2 \\
2 & 2 & & & 2 \\
4 & 2 & & & 2 \\
4 & 4 & & & \\
 & & 1 & & \\
 & & & 1 & \\
 & & & & 1
\end{bmatrix}
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}
\geq
\begin{bmatrix}
300 \\ 300 \\ 300 \\ 300 \\ 300 \\ 1152 \\ 1152 \\ 1152 \\ 1452 \\ 1452 \\ 1452 \\ 1452 \\ 1452 \\ 1452 \\ 1568 \\ 1568 \\ 1568 \\ 1568 \\ 1868 \\ 1868 \\ 1868 \\ 1868 \\ 3020 \\ 3020 \\ 70.6 \\ 138.1 \\ 92.8
\end{bmatrix}
$$

FIGURE 20. LINEAR RESTRICTIONS FOR EXAMPLE 3



Roof Load $= \dfrac{50 \text{ lb/sq.ft} \times 20' \times 1.85}{1000}$

$= 1.85^{k}/\text{lin. ft.}$

Floor Load $= \dfrac{100 \text{ lb./sq.ft.} \times 20' \times 1.85}{1000}$

$= 3.70^{k}/\text{lin. ft.}$

Frames are 20' on center

Load Factor = 1.85

FIGURE 21. GIVEN CONDITIONS FOR EXAMPLE 4

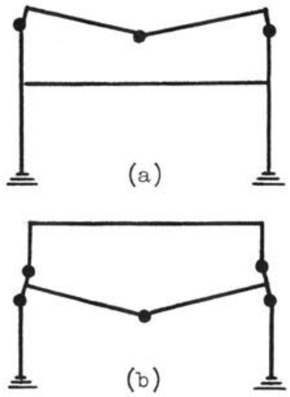


FIGURE 22. MECHANISMS FOR EXAMPLE 4

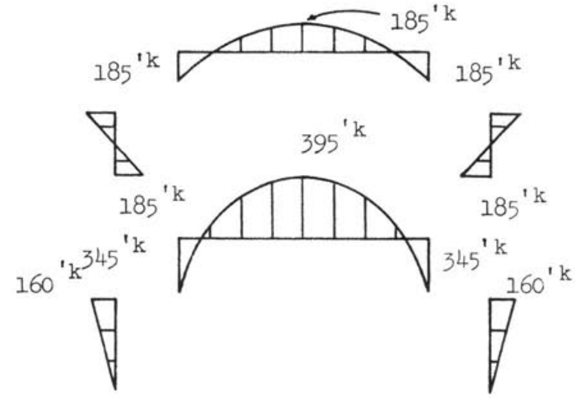
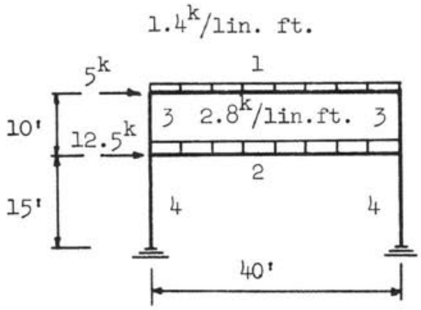


FIGURE 23. MOMENT DIAGRAM FOR EXAMPLE 4



Roof Load = $\frac{50 \text{ lb./sq.ft.x20'x1.4}}{1000}$ = 1.4^k/lin.ft.

Floor Load = $\frac{100 \text{ lb./sq.ft.x20'x1.4}}{1000}$

= 2.8^k/lin.ft.

Wind Velocity 108 MPH By A.S.C.E. Wind Load = 35.7 lb./sq.ft.

Wind Force Roof = $\frac{35.7 \times 20'\text{x}5'\text{x}1.4}{1000}$ = 5^k

Wind Force 2nd Floor = $\frac{35.7 \times 20 \times 12.5 \times 1.4}{1000}$

= 12.5^k

Frames are 20' on center

Load Factor = 1.4

FIGURE 24. GIVEN CONDITIONS FOR EXAMPLE 5

Max. Positive Moment = 147.8^{1k}
at 0.62' left of ℄

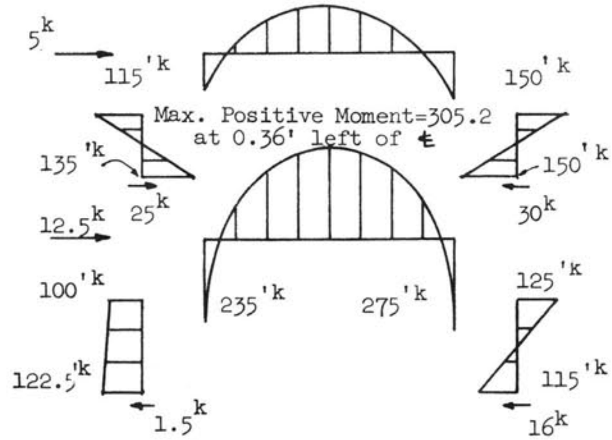


FIGURE 25. MOMENT DIAGRAM AND COLUMN SHEARS FOR EXAMPLE 5

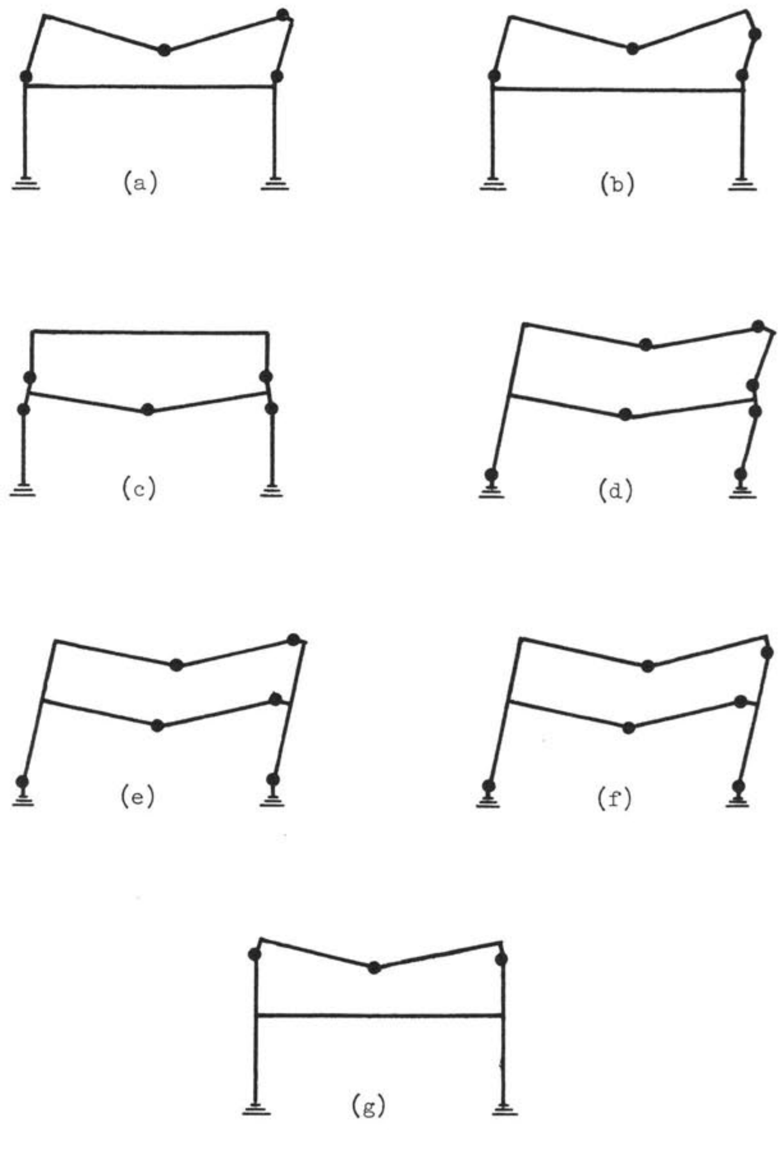


FIGURE 26. MECHANISMS FOR DIAGRAM 5

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TABLE 1.

INPUT TO COMPUTER PROGRAM: PROPERTIES OF SECTIONS

No.	Nominal Depth	Weight Per Ft.	M_p	Area	r_x
1	6.00	4.40	7.80	1.30	2.37
2	7.00	5.50	11.10	1.61	2.74
3	8.00	6.50	15.00	1.92	3.12
4	10.00	9.00	25.40	2.64	3.85
5	12.00	11.80	39.10	3.45	4.57
6	10.00	15.00	43.90	4.40	3.95
7	12.00	16.50	56.70	4.86	4.65
8	14.00	17.20	67.10	5.05	5.40
9	12.00	22.00	80.70	6.47	4.91
10	12.00	27.00	104.40	7.97	5.06
11	14.00	30.00	129.50	8.81	5.73
12	14.00	34.00	149.90	10.00	5.83
13	16.00	36.00	175.70	10.59	6.49
14	16.00	40.00	200.00	11.77	6.62
15	16.00	45.00	226.00	13.24	6.64
16	18.00	50.00	277.00	14.71	7.38
17	18.00	55.00	307.00	16.19	7.41
18	18.00	60.00	337.00	17.64	7.47
19	21.00	62.00	396.00	18.23	8.53
20	21.00	68.00	439.00	20.02	8.59
21	21.00	73.00	473.00	21.46	8.64
22	24.00	76.00	550.00	22.37	9.68
23	24.00	84.00	616.00	24.71	9.78
24	27.00	94.00	764.00	27.65	10.87
25	27.00	102.00	837.00	30.01	10.96
26	30.00	108.00	950.00	31.77	11.85
27	30.00	116.00	1038.00	34.13	12.00
28	30.00	124.00	1120.00	36.45	12.11
29	33.00	130.00	1282.00	38.26	13.23
30	33.00	141.00	1411.00	41.51	13.39
31	36.00	150.00	1594.00	44.16	14.29
32	36.00	160.00	1714.00	47.09	14.38
33	36.00	170.00	1833.00	49.98	14.47
34	36.00	182.00	1972.00	53.54	14.52
35	36.00	194.00	2110.00	57.11	14.56
36	33.00	220.00	2300.00	64.73	13.79
37	36.00	230.00	2590.00	67.73	14.88
38	36.00	245.00	2770.00	72.03	14.95
39	36.00	260.00	2960.00	76.56	15.00
40	36.00	280.00	3210.00	82.32	15.12
41	36.00	300.00	3450.00	88.17	15.17

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design will allow for a moment diagram which nowhere exceeds the moment capacities of the members. But even the experienced designer would know the location of the intermediate hinges for only a very limited number of load distributions and end moment combinations.

Two courses of action are available to meet this problem. First, the probable region of hinge location for each beam might be bracketed by including two (or even more) mechanism equations in the array of linear restrictions. However, for each additional basic beam equation considered, the total possible number of combinations of basic equations would be doubled. Because of this fact, then, this course of action is not recommended except for frames composed of only a few members.

The second course of action takes account of the fact that the moment capacities provided by the minimum-weight design using standard sections will usually exceed slightly the precise moments dictated by the loads. Therefore, only one basic equation is included in the linear restrictions for each beam in the frame, with the intermediate hinge positioned by experience. After the minimum-weight design has been determined, each beam is checked to determine whether or not its moment capacity is exceeded at any point. It is believed that in the majority of problems the moment capacities of the minimum-weight design will not be exceeded. However, if the moment capacity of any member is exceeded, a procedure similar to that outlined in Section III D should be followed. This course of action is illustrated in Examples 3 and 5.

G. MINIMUM-WEIGHT PROBLEM WITH RESPECT TO TWO OR MORE LOAD SYSTEMS

In most situations the designer must provide a structure capable of withstanding more than one system of loading. Such situations may arise in the case of "checkerboard" loading or where, as in the case of wind and other lateral forces, reduced load factors are usually prescribed for combinations of such forces with gravity loads.

The frame which is to be designed for two different load systems will now be considered. Let Φ_I and Φ_{II}, each consisting of n sections for the n members in the frame, be the minimum-weight solutions for load systems I and II, respectively. These solutions satisfy, respectively, the requirements imposed by column axial forces in load systems I and II, and, respectively, the moment requirements expressed by the following systems of inequalities:

$$\left[A_I\right] \quad \left\{X_I\right\} \geq \left\{b_I\right\} \tag{3.17}$$

$$\left[A_{II}\right] \quad \left\{X_{II}\right\} \geq \left\{b_{II}\right\} \tag{3.18}$$

In many cases $\left[A_I\right] \equiv \left[A_{II}\right]$. However, in the general case $\left[A_I\right] \not\equiv \left[A_{II}\right]$.

Let FW_I and FW_{II} be the frame weights, respectively, for the two minimum-weight solutions.

The following possibilities must be considered:

$$\Phi_{Ij} \equiv \Phi_{IIj} \qquad j = 1,2,\dots.n \tag{3.19}$$

$$\Phi_{Ij} \geq \Phi_{IIj} \qquad j = 1,2,\dots.n \tag{3.20}$$

$$\Phi_{Ij} \leq \Phi_{IIj} \qquad j = 1,2,\dots.n \tag{3.21}$$

$$\left.\begin{array}{ll} \Phi_{Ii} \equiv \Phi_{IIi} & i = 1,2,\dots.r \\ \Phi_{Ij} \leq \Phi_{IIj} & j = r+1,r+2,\dots n \end{array}\right\} \tag{3.22}$$

$$\left.\begin{array}{ll} \Phi_{Ii} \geq \Phi_{IIi} & i = 1,2,\dots.r \\ \Phi_{Ij} \leq \Phi_{IIj} & j = r+1,r+2,\dots s \\ \Phi_{Ik} \equiv \Phi_{IIk} & k = s+1,s+2,\dots n \end{array}\right\} \tag{3.23}$$

If Equation (3.19) is satisfied, all members of Φ_I are identical to the corresponding members of Φ_{II}, and the minimum-weight solution is determined. If Equation (3.20) is satisfied, all members of Φ_I exceed in size their counterparts in Φ_{II}, and the minimum-weight solution is Φ_I. Likewise, if Equation (3.21) is satisfied, the minimum-weight solution is Φ_{II}. If Equation (3.22) is satisfied, r members of Φ_I and Φ_{II} are identical, while $(n-r)$ members of Φ_{II} exceed in size their counterparts in Φ_I. In this case, the minimum-weight solution is Φ_{II}. Of course, the minimum-weight solution is Φ_I if the sign of the inequality is reversed.

If Equation (3.23) is satisfied, r members of Φ_I exceed in size the corresponding members of Φ_{II}, $(s-r)$ members of Φ_{II} exceed in size the corresponding members of Φ_I, and $(n-s)$ members are identical. In this case, a minimum-weight solution has not necessarily been determined. Suppose $FW_I > FW_{II}$. FW_I, then, becomes a lower bound to the minimum-weight solution. It is quite possible, and not uncommon, for the moment capacities represented by Φ_I to satisfy both Equations (3.17) and (3.18) and the most restrictive axial loading from both systems. Should this happen, Φ_I is also an upper bound, and the minimum-weight solution is Φ_I. More generally, an upper bound to the minimum-weight solution is obtained with a new design $\Phi_{III} = \Phi_{Ii} + \Phi_{IIj} + \Phi_{Ik}$ whose weight is FW_{III}. If the difference in weight between FW_{III} and the lower bound FW_I is considered to be small enough, Φ_{III} may be selected as the final solution. If this difference is not acceptable, the following procedure may be initiated. A suitable bandwidth for the members of Φ_I is determined. The lightest design within this bandwidth which satisfies both Equation (3.17) with axial loads from load system I and Equation (3.18) with axial loads from load system II becomes the minimum-weight solution, within the limitations imposed by the prescribed bandwidths.

Should the structure be subjected to N load systems, the minimum-weight design with respect to two load systems would be compared with the minimum-weight design for a third load system. This process would be continued until all N load systems had been included in the analysis. It would, of course, be expedient to make an effort to select the two most dominant load systems for the first comparison.

The minimum-weight problem with respect to two or more load systems may be approached in another manner by constructing a solution space bounded by the most restrictive equations from each load-system set. However, if the most restrictive axial loads are taken for the columns, and these restrictions are used in conjunction with the composite solution space, an upper bound to the minimum weight will be obtained. This is because, in some members, we may be combining the axial load P from one load system with the bending moment M_0 from another. This upper bound occasionally may differ appreciably from the minimum-weight solution, but it is the author's opinion that for most practical load systems it will be a close approximation.

H. ADDITIONAL CONSIDERATIONS

The accommodation of the bending moments and the axial forces induced in a frame by applied loading usually serves as the primary criterion for the design of the structure. After the individual members have been selected, problems of deflection, incremental collapse, cyclic loading, connections, clearance, etc., may need to be considered. Upon checking the adequacy of the minimum-weight design against these so-called secondary criteria, it may be found necessary to change one or more members. In this case, the designer may consider the changed members to be constants in the solution, and repeat the process as follows: The corresponding variables are dropped from the objective function, and the moment capacities M_p (and M_0) of the changed members are substituted for the corresponding variables in the array of linear restrictions. This change results in fewer variables, and, usually, a reduction in the number of equations. Of course, the new design will be found to have a greater weight.

Another problem of this type arises when, either initially or subsequent to the first solution, practical considerations suggest continuity of size in two or more members adjacent to one another. This situation presents very little difficulty in the case of the beam. All that need be done is to represent the M_p of all such sections by the same variable X in the array of linear restrictions. Of course, the total length of all members joined under one variable must be used in determining the corresponding objective function coefficient. The section determined by the solution process for this variable is then used for all members joined under the variable.

A similar provision for a column that is to be continuous in size across one or more joints presents much more of a problem. Because there may be significant differences in axial loads and lengths of adjacent-story columns as well as in the probable range of M_O, the weight-moment relationships could be quite different. Therefore, the definition of the corresponding objective function coefficient becomes arbitrary. Furthermore, the rotations of the hinges of these members must be separated in all linear restrictions for the trial portion of the solution because, for a given section, each of these joined members will generally have a different capacity M_O. The solution process should be initiated with the columns to be joined represented by different variables. After the so-called secondary criteria mentioned above have been satisfied for all members, a section is selected for the joined columns. The allowable value of M_O is now calculated for each column. The M_O values are substituted for the corresponding variables in the array of linear restrictions, and the solution is completed as outlined in the first part of this section.

In addition to the primary requirements for columns discussed in Section III C, the AISC Specification imposes the following restrictions:

$$L/r_x \leq 120 \qquad (3.24)$$

$$P/P_y \leq 0.6 \qquad (3.25)$$

Furthermore, premature buckling of the webs of I and WF shapes must be precluded in the region of a plastic hinge. This requirement is met if the following equation is satisfied:

$$d/w \leq 70 - 100\ P/P_y \qquad (3.26)$$

except that d/w need not be less than 43. In this equation, d = depth of section and w = thickness of web. Equation 3.26 also applies to beams. However, since $d/w \leq 70$ for all standard rolled shapes and the axial compression P is usually relatively small for beams, in this paper the restriction is omitted for beams.

A column which does not satisfy Equation 3.26 may be used if its web is reinforced with longitudinal stiffeners in the plastic-hinge zones. Therefore, the minimum-weight problem can be approached in either of two ways. Equation 3.26 can be used at the outset to establish a minimum section for each column; this procedure determines a lower limit on M_O just as in the case of Equation 3.8. Alternatively, this step may be omitted and the locations, if any, of required stiffening determined in the resulting minimum-weight design.

Since Equation 3.26 tends to require sections with relatively stocky webs where axial loads are relatively large, occasionally the minimum-weight design may contain columns which are not economy shapes. This possibility was not considered in this paper. ●

IV. EXAMPLES

A. GENERAL REMARKS

The examples in this section are based on A7 Steel (σ_y = 33 ksi). Provisions of the AISC Specification discussed in Section III C and D have been adhered to, except that Equation 3.26 was not enforced. Therefore, some of the resulting frames contain columns which would require longitudinal web stiffeners in the plastic-hinge zones. Of course, Equation 3.26 could be imposed as an additional constraint to obtain a frame that would require no stiffeners, so that the relative economy of the two alternatives could be determined. The computer program described in Section IV C was used for Examples 3, 4, and 5, using the IBM 7090.

B. EXAMPLES

1. The given conditions for this example are shown in Figure 16. The example will be solved first by the linear programming method, but without making allowance for column action. As has been done by a number of investigators, a single linear relationship between weight per foot and M_p will be assumed to hold for the entire range of standard sections. The objective function coefficient, then, will consist of only the length of the corresponding members. This solution will subsequently be compared with the more precise solution outlined in this study.

A more general method of determining the first basic feasible solution is also described in this example. It will be recalled from Section II C that, for a basic feasible solution to exist, m+n linearly independent vectors must be in solution. This requirement is satisfied if the m+n vectors of the solution form an identity matrix. Instead of solving for an initial basic feasible solution, we may assume an entirely artificial one. To do this, we simply add to our augmented matrix an identity matrix consisting of the same number of new variables as we have equations. These new variables must be included in the objective function. However, we assign them such arbitrarily large coefficients as to drive them from solution. The final solution is not valid unless all these artificial variables are absent. Further explanation and proof of validity of this technique may be found in Reference 10.

The statement of the problem is as follows:

Minimize

$$F'_w = 40X_1 + 32X_2$$

subject to

$$4X_1 \geq 1000$$

$$2X_1 + 2X_2 \geq 1000$$

where X_1 = plastic moment required for beam and
X_2 = plastic moment required for each column

The first equation is the objective (weight) function. The first inequality derives from the beam mechanism in which all three hinges are in the beam, while the second derives from the same source, but with one hinge in the beam and one in each column top. This completes the list of mechanisms for this frame.

The addition of slack variables X_3 and X_4 produces the following result:

Minimize

$$F'_w = 40X_1 + 32X_2 + 0X_3 + 0X_4$$

subject to

$$4X_1 - X_3 = 1000$$

$$2X_1 + 2X_2 - X_4 = 1000$$

Using the technique mentioned above for obtaining the first basic feasible solution, we introduce the artificial variables X_5 and X_6, incorporating them in the objective function with arbitrary (large) coefficients of 500.

Minimize

$$F'_w = 40X_1 + 32X_2 + 0X_3 + 0X_4 + 500X_5 + 500X_6$$

Subject to

$$4X_1 - X_3 + X_5 = 1000$$

$$2X_1 + 2X_2 - X_4 + X_6 = 1000$$

In tableau form, this is written

		40	32	0	0	500	500		
	Basis	$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	$\overline{P}_5$	$\overline{P}_6$	$\overline{P}_0$	θ
500	$\overline{P}_5$	4	0	-1	0	1	0	1000	
500	$\overline{P}_6$	2	2	0	-1	0	1	1000	
(F_j-C_j)									

Values of F_j-C_j are determined next. Thus, to get F_1-C_1, we have

$$F_1-C_1 = 500 \times 4 + 500 \times 2 - 40 = 2960$$

After this step, the tableau appears thus:

		40	32	0	0	500	500		
	Basis	$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	$\overline{P}_5$	$\overline{P}_6$	$\overline{P}_0$	θ
500	$\overline{P}_5$	4	0	-1	0	1	0	1000	
500	$\overline{P}_6$	2	2	0	-1	0	1	1000	
(F_j-C_j)		2960	968	-500	-500	0	0		

This solution is not optimum because not all $F_j-C_j \leq 0$. Therefore, we compute values of θ by dividing each element in the P_0 column by the element which is in its row and in the column containing the largest F_j-C_j. The tableau appears then in the following form:

		40	32	0	0	500	500		
	Basis	$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	$\overline{P}_5$	$\overline{P}_6$	$\overline{P}_0$	θ
500	$\overline{P}_5$	4	0	-1	0	1	0	1000	250
500	$\overline{P}_6$	2	2	0	-1	0	1	1000	500
(F_j-C_j)		2960	968	-500	-500	0	0		

Since vector $\overline{P}_5$ corresponds to the smallest positive θ, it goes out of solution while $\overline{P}_1$, corresponding to the largest F_j-C_j, comes in. This transfer is accomplished by dividing the first row by four, and then adding minus two times the modified first row to the second row. The result, after computation of F_j-C_j and θ, is

		40	32	0	0	500	500		
	Basis	$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	$\overline{P}_5$	$\overline{P}_6$	$\overline{P}_0$	θ
40	$\overline{P}_1$	1	0	-1/4	0	1/4	0	250	∞
500	$\overline{P}_6$	0	2	1/2	-1	-1/2	1	500	250
$(F_j$-$C_j)$		0	968	240	-500	-740	0		

Again, not all F_j-$C_j \leq 0$. The replacement of $\overline{P}_6$ by $\overline{P}_2$ is accomplished by dividing the second row by 2. The new tableau, with values of F_j-C_j computed, is

		40	32	0	0	500	500	
	Basis	$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	$\overline{P}_5$	$\overline{P}_6$	$\overline{P}_0$
40	$\overline{P}_1$	1	0	-1/4	0	1/4	0	250
32	$\overline{P}_2$	0	1	1/4	-1/2	-1/4	1/2	250
$(F_j$-$C_j)$		0	0	-2	-16	-498	-484	

Since all F_j-$C_j \leq 0$, we have obtained the minimum-weight solution, for which $X_1 = X_2 = 250$ foot-kips. The initial design selected on the basis of these theoretical moments is an 18 WF50 for both beam and columns. The selection results in a frame weight of 3600 lbs.

The following designs were determined by increasing the beam one section in the economy table for each succeeding design and selecting the smallest section for the column which satisfies the mechanism inequalities. All columns also satisfy the Case II column formula.

Beam	M_p(foot-kips)	M_0(foot-kips)	Column	Frame Weight (lbs.)
18WF50	277	226	16WF45	3440
18WF55	307	200	16WF40	3480
18WF60	337	175	16WF36	3552
21WF62	396	104	12WF27	3344
21WF73	473	34	12Jr.11.8	3298
24WF76	550	2	6Jr.4.4	3181

Note that six additional trials were required to determine the minimum-weight design.

This example will now be solved using the more realistic moment-weight relationships and taking into account the effect of axial loads on the columns.

The range in M_p which must be considered for the beam is $M_p(\text{min.}) = PL/8 = 250$ foot-kips to $M_p(\text{max.}) = PL/4 = 500$ foot-kips. The method of averages was used for all economy sections within this range to determine the linear relationship

$$D_1 = 24.8 + 0.098M_{P1} \quad (4.1)$$

The possible range in M_0 which must be considered for the columns is M_0(min.) = 0 to M_0(max.) = 250 foot-kips. Since the axial force in the column is 25 kips (Figure 16), values of M_0 for an axial load 25 kips and a column height 16 feet were calculated in accordance with Equation (3.4) for all economy sections within this range. The method of averages was used to determine the linear relationship

$$D_2 = 5.5 + 0.178M_{02} \quad (4.2)$$

The objective function is now written in accordance with Equation (3.12)

$$F'_w = (40)(0.098)X_1 + (16{+}16)(0.178)X_2$$
$$4X_1 + 6X_2 \quad (4.3)$$

In general, these coefficients should not be rounded.

The first basic feasible solution in tableau form with coefficients of the artificial variables taken to be 100 is

		4	6	0	0	100	100		
		$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	$\overline{P}_5$	$\overline{P}_6$	$\overline{P}_0$	θ
100	P_5	4	0	-1	0	1	0	1000	250
100	P_6	2	2	0	-1	0	1	1000	500
(F_j-C_j)		596	194	-100	-100	0	0		

Replace $\overline{P}_5$ by $\overline{P}_1$:

		4	6	0	0	100	100		
		$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	$\overline{P}_5$	$\overline{P}_6$	$\overline{P}_0$	θ
4	P_1	1	0	-1/4	0	1/4	0	250	∞
100	P_6	0	2	1/2	-1	-1/2	1	500	250
(F_j-C_j)		0	194	49	-100	-149	-		

Replace $\overline{P}_6$ by $\overline{P}_2$:

		4	6	0	0	100	100		
		$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	$\overline{P}_4$	$\overline{P}_5$	$\overline{P}_6$	$\overline{P}_0$	θ
4	P_1	1	0	-1/4	0	1/4	0	250	-250/4
6	P_2	0	1	1/4	-1/2	-1/4	1/2	250	250/4
(F_j-C_j)		0	0	1/2	-3	-100.5	-97		

Replace $\overline{P}_2$ by $\overline{P}_3$:

		4	6	0	0	100	100		
		$\overline{P}_1$	$\overline{P}_2$	$\overline{P}_3$	4	$\overline{P}_5$	$\overline{P}_6$	$\overline{P}_0$	θ
4	P_1	1	1	0	-1/2	0	1/2	500	
0	P_3	0	4	1	-2	-1	2	1000	
(F_j-C_j)		0	-2	0	-2	-100	-98		

This is the theoretical minimum-weight solution, for which $X_1 = 500$ foot-kips and $X_2 = 0$. The initial design, selected on the basis of these theoretical moments and an axial load of 25 kips for the columns, is a 24WF76 for the beam and a 6 Jr.4.4 for the columns, which agrees with the solution found previously by trial. Of course, this is not a practical solution, but it does suggest that the column be no larger than practical considerations require.

2. Example 2 will be defined by removing the brace against sidesway in Example 1, but without making any changes in loading or geometry. We must add an inequality to provide for the restriction imposed by sidesway of the frame. For an axial load of 25 kips and a column height of 16 feet, Equation (3.8) dictates a 12B16.5 as the minimum section. Equation (3.4) is now used to calculate the maximum value of M_0 that may be used in conjunction with the 25-kip axial load for this section. The result is $M_0(\text{min.}) = 54$ foot-kips. Therefore, the following inequality is added to the two mechanism inequalities:

$$X_2 \geq 54 \qquad (4.4)$$

The possible range in M_0 which must be considered for the columns is $M_0(\text{min.}) = 54$ foot-kips to $M_0(\text{max.}) = 250$ foot-kips. The method of averages is used to determine the linear relationship for all economy sections within this range:

$$D_2 = 9.1 + 0.155\, M_{02} \qquad (4.5)$$

Note that the constant term in this equation is greater than the constant term in Equation (4.2), while the slope is less, because this line segment does not include the steeper portion of the weight-moment plot near the origin that was included in the determination of Equation (4.2). Equation (4.1) for the relationship between weight and moment for the beam remains the same.

The objective function, without slack and artificial variables, is

$$F'_w = (40)(0.098)X_1 + (16{+}16)(0.155)X_2$$
$$\approx 4X_1 + 5X_2 \qquad (4.6)$$

The solution is shown in Figure 17. The theoretical minimum-weight solution yields $X_1 = 446$, $X_2 = 54$.

The initial design, selected on the basis of these theoretical moments and an axial load of 25 kips on the columns, is a 21WF73 for the beam and a 12B16.5 for the columns. This design results in a frame weight of 3448 lbs. Further trials in this region of theoretical moments show that a 21WF68 for the beam and a 14B17.2 for the columns is a feasible solution. The resulting frame weights 3270 lbs., which was found to be the minimum weight for this example. It is to be noted that the sections comprising the minimum weight design are only one section removed (Table 1) from the sections comprising the initial solution. The difference in weight is about five per cent.

C. DESCRIPTION OF COMPUTER PROGRAM

The computer program which was developed to determine the minimum-weight design of frames is described here briefly. The flow chart is shown in the Appendix.

1. The input consists of the following data:

a. The nominal depth and weight per foot, the plastic moment capacity M_p, the area, and the radius of gyration r_x of the standard "economy" sections (Table 1).
b. The array of mechanism inequalities.
c. The axial load for each column.
d. The lengths of all members.
e. The range in M_p to be considered for each beam and the range in M_0 to be considered for each column.
f. The bandwidth to be surveyed for each member for the trial portion of the program.

g. The condition of the frame (braced or unbraced) with respect to sidesway.

2. The solution is accomplished in the following steps:

a. Linear relationships between M_p and the weight per foot for each beam, and between M_0 and the weight per foot for each column, are determined. With these relationships and the individual member lengths, the objective function is compiled.

b. If the frame is subject to sidesway, a linear restriction is added for each column specifying the proper lower limit for M_0.

c. The inequalities are augmented by the slack and artificial variables to form the first basic feasible solution as an array of equalities.

d. The solution is effected by the simplex method, yielding the theoretical minimum-weight moments.

e. From these moments and the axial-load input for columns, the lightest section is selected for each member from the list furnished in the input data. This selection becomes the initial solution.

f. The input bandwidth is positioned with respect to the initial solution, and the feasibility of all designs within this bandwidth are determined. The feasible designs are compared to determine the least-weight design.

g. Each inequality is converted to an equality to determine the smallest proportionate increase in load required to produce collapse. The ratio of this smallest proportionate increase to the factored service load is called the load-increase factor.

3. The output consists of the following information:

a. All input data are output for problem identification and reference.

b. Linear relationships between weight per foot and M_p for beams and M_0 for columns. The average residual and maximum residual between the linear relationships and the true values is also output as an indication of the reliability of the linear relationships.

c. The objective function.

d. The theoretical moments and the initial design.

e. The least-weight design within the bandwidths specified.

f. The load-increase factor and the equation of the corresponding mechanism of collapse.

g. The number of cycles required by the simplex procedure.

D. ADDITIONAL EXAMPLES

3. The given conditions for this example are shown in Figure 18. The sections and section-properties input are shown in Table 1. The array of inequalities input is shown in Figure 20. The axial loads and the range in M_p and M_0 input for each member are shown at the top of the next page.

The bandwidth was set wide enough to include the full range of probable values for M_p and M_0. This range can be covered with a problem this limited in size, thus insuring the determination of the minimum-weight solution.

The linear relationships between weight per foot and moment capacity, with average and maximum residuals, expressed in the form of Equations (3.3) and (3.9), are shown on the next page.

The resulting objective function, without slack and artificial variables, is

$$F'_w = 4.54X_1 + 5.23X_2 + 2.02X_3 + 2.03X_4 + 2.06X_5$$

Member	Axial Load	$M_{p(min.)}$	$M_{p(max.)}$	$M_{0(min.)}$	$M_{0(max.)}$
1	0	288	576		
2	0	392	784		
3	48			0*	288
4	104			0*	288
5	56			0*	392

Member	a_k	b_k	a_i'	b_i'	Average Residual	Maximum Residual
1	26.3	0.095			1.22	2.30
2	26.4	0.093			1.70	3.72
3			13.8	0.135	0.97	2.51
4			18.0	0.135	0.89	1.84
5			13.7	0.137	0.82	1.95

The theoretical moments, in foot-kips, from the simplex solution and the corresponding initial design are

Member	Theoretical Moment	Initial Design	Actual Moment M_p	M_0
1	363	21WF62	396	
2	392	21WF62	396	
3	70.6	12B22	80.7	70.6
4	138.1	16WF36	175.7	138.1
5	392	21WF62	396	396**

Investigation during the trial portion of the program of all possible variations of the initial solution, over the bandwidth specified, showed that the initial solution is, in fact, the minimum-weight design.

Equation (25) was found to be satisfied exactly by the initial (minimum-weight) solution. Therefore the load-increase factor is zero.

Eight cycles of the simplex method were required in this solution. Twenty-four seconds of machine time were required for the complete solution.

In the formulation of the basic beam equations the intermediate hinges were assumed to form in the center of each beam. Because these hinges will form off center for certain combined mechanisms, the minimum-weight solution is checked against the yield criterion (Section III F). An admissible moment diagram is shown in Figure 19.

4. The two-story symmetrical frame shown in Figure 21 is solved in this example. The frame is subjected to only gravity loading, and is considered to be unbraced against sidesway. The combination of wind and reduced gravity load will be considered in Example 5. The inequalities to be satisfied consist of the mechanism inequalities and the requirements of Equation (3.8):

*Since the frame is unbraced, these limits are changed by the computer to those shown in Equations (25), (26), and (27) of Figure 20.

**Galambos and Ketter[9] have shown that if $P/P_y \leq 0.15$, M_0 is negligibly smaller than the fully plastic moment M_p.

$$\begin{array}{c} (1)\\(2)\\(3)\\(4)\\(5)\\(6) \end{array} \quad \begin{bmatrix} 4 & & & \\ & 4 & & \\ 2 & & 2 & \\ & 2 & 2 & 2 \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \geq \begin{bmatrix} 740 \\ 1480 \\ 740 \\ 1480 \\ 36.8 \\ 160.0 \end{bmatrix}$$

The first two equations are basic beam equations. The next two are combinations of beam and joint equations (Figure 22a and b, respectively), while the last two are the lower limits of M_0 dictated by the sidesway condition for the columns. It should be noted that the panel mechanism was not used in any combination, because the frame and loading are symmetrical. Thus, the complete array of inequalities has been established.

The axial loads and the range in M_p and M_0 input for each member are

Member	Axial Load	$M_{p(min.)}$	$M_{p(max.)}$	$M_{0(min.)}$	$M_{0(max.)}$
1	0	185	370		
2	0	370	740		
3	37			0	185
4	111			0	370

Again, the bandwidth was set wide enough to include the full range of probable M_p and M_0.

The linear relationships between weight per foot and moment capacity, with average and maximum residuals, expressed in the form of Equations (3.3) and (3.9), are

Member	a_k	b_k	a_i'	b_i'	Average Residual	Maximum Residual
1	9.0	0.153			1.16	1.92
2	29.0	0.089			1.30	2.12
3			8.1	0.175	1.06	3.29
4			24.8	0.107	1.42	2.70

The resulting objective function, without slack and artificial variables, is

$$F_w' = 6.14X_1 + 3.55X_2 + 3.50X_3 + 3.22X_4$$

The theoretical moments in foot-kips, from the simplex solution, and the corresponding initial design are

Member	Theoretical Moment	Initial Design	Actual Moment M_p	Actual Moment M_0
1	185	16WF40	200	
2	370	21WF62	396	
3	210	16WF45	226	226
4	160	16WF40	200	160

The minimum-weight design determined in the trial solution was the same as the initial design, except that member three was reduced to a 16WF40.

Since Equation 6 is satisfied with no increase in loading, the load-increase factor is zero.

Seven cycles of the simplex method were used in this solution. Twelve seconds of machine time were used for the complete solution.

The distribution of moments given in Figure 23 shows that the yield criterion is satisfied. However, this check is not required since the minimum-weight solution was based on the complete array of restrictions.

5. This example illustrates the determination of a minimum-weight frame based on a reduced array of inequalities, as discussed in Section IIID, together with the subsequent check of the yield criterion and procedures which may be used for revising the design when the criterion is not satisfied.

The geometry of the frame (Figure 24) is the same as that of Example 4. The loads to be supported are the gravity loads of that example plus a wind load of 36 psf. The required load factor is 1.4. The solution will be based on the following partial array of inequalities:

$$\begin{array}{c} (1)\\(2)\\(3)\\(4)\\(5)\\(6)\\(7)\\(8)\\(9)\\(10)\\(11)\\(12) \end{array} \begin{bmatrix} 4 & & & \\ & 4 & & \\ & & 4 & \\ & & & 4 \\ 4 & & 2 & \\ 2 & & 4 & \\ & 2 & 2 & 2 \\ 4 & 2 & 2 & 4 \\ 4 & 4 & & 2 \\ 2 & 4 & 2 & 2 \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \geq \begin{bmatrix} 560 \\ 1120 \\ 50 \\ 263 \\ 610 \\ 610 \\ 1120 \\ 1993 \\ 1993 \\ 1993 \\ 33.2 \\ 125.6 \end{bmatrix}$$

Equations (1) and (2) are beam equations. Equations (3) and (4) are panel equations, while Equations (5) through (10) are combinations of beam, joint, and panel equations (Figure 26 a, b, c, d, e, and f, respectively). Equations (11) and (12) are the lower limits on M_0, required by the sidesway restrictions, for the columns.

The remaining input and output data are similar in format to those of Example 4, and are given below without further explanation.

Member	Axial Load	$M_{p(min.)}$	$M_{p(max.)}$	$M_{0(min.)}$	$M_{0(max.)}$
1	0	140	280		
2	0	280	560		
3	28			0	140
4	84			0	280

Member	a_k	b_k	a_i'	b_i'	Average Residual	Maximum Residual
1	10.2	0.151			1.08	2.07
2	26.8	0.094			1.28	2.45
3			5.7	0.201	1.03	3.03
4			15.6	0.140	0.89	1.80

$$F_W' = 6.05X_1 + 3.75X_2 + 4.02X_3 + 4.20X_4$$

The theoretical moments in foot-kips from the simplex solution and the initial design, are

Member	Theoretical Moment	Initial Design	Actual Moment M_p	M_0
1	140	14WF34	150	
2	321	18WF60	337	
3	82.5	12WF27	104	104
4	156.5	16WF40	200	177

The minimum-weight design determined in the trial solution is the same as the initial design except that member 2 is reduced to an 18WF55, and member 4 to a 16WF36. The weight of the frame based upon this design is 5180 lbs.

Because a reduced array of linear restrictions was considered, the moments provided by the minimum-weight design must be checked against the yield criterion. An attempt to determine a closure of the moment diagram revealed that either the moment at the center of the roof beam or the moments at the tops of the upper-story columns exceeded the moments furnished by the minimum-weight design for these members.

As was discussed in Section III D there are two ways in which the solution may be corrected. First, consideration is given to increasing the size of the roof beam or the upper-story columns or both, so as to satisfy the yield criterion. A few trials indicated an increase in the upper-story columns (member three) to 14WF34 sections as the solution which results in the least increase in weight. This design satisfies the yield criterion with the intermediate beam hinges of the collapse mechanism correctly located left of center. The weight of the frame based upon this design is 5320 lbs.

The true minimum-weight solution is now known to lie between the lower bound, 5180 lbs., and the upper bound, 5320 lbs. This is a difference of only 2.7 per cent, so that the upper-bound design might be accepted as the solution.

The second way in which the solution may be completed is to increase the array of linear restrictions, with the intention of including restrictions which form the basis of the true minimum-weight solution and which were omitted in the original array. Because the yield criterion was not satisfied in the region of the roof beam and its included joints, mechanisms in this region will be investigated. The most obvious restriction missing in the original array is the combination of beam and joint mechanisms (Figure 26g) which results in the inequality

$$2X_1 + 2X_3 \geq 560$$

Although additional restrictions may be sought and added to the array of inequalities, the example will now be solved with the addition of the single inequality above.

The new theoretical moments in foot-kips, from the simplex solution and the new initial design, are

Member	Theoretical Moment	Initial Design	Actual Moments M_p	M_0
1	140	14WF34	150	
2	285.5	18WF55	307	
3	140	14WF34	150	150
4	145.5	16WF36	175	150

The new minimum-weight design determined in the trial solution is the same as the new initial design, except that member four is reduced to a 14FW34. The weight of the frame based upon this design is 5260 lbs., which is, of course, between the upper (5320 lbs.) and lower (5180 lbs.) bounds determined previously.

Again, this design must be checked against the yield criterion because the solution is based upon a reduced array of linear restrictions. The distribution of moments given in Figure 25 shows the yield criterion to be satisfied. The reduced array, then, must contain the basis of solution, and the minimum weight has been determined. The following comparison of the designs obtained by the two methods of correcting for the (inadvertent) omission of a controlling mechanism in the original array of inequalities is of interest:

Member	First Revision	Second Revision
1	14WF34	14WF34
2	18WF55	18WF55
3	14WF34	14WF34
4	16WF36	14WF34
Weight	5320 lbs.	5260 lbs.

V. SUMMARY

Methods for finding the minimum weight of plastically designed steel frames which are to be composed of standard sections have been based on the assumption that the weight-moment relationships of the spectrum of standard sections can be approximated by a continuous function. Some investigators have used a best-fit, nonlinear function, others a best-fit linear function. The first method of solution of the frame in Example 1 illustrates the latter method. It is interesting to note that the result is a frame which is 13 per cent heavier than that found in the same example using methods proposed in this thesis.

The plastic moment which can be carried by a beam-column is less than the fully plastic moment in pure bending, owing to the axial compressive force. This reduction depends on the relative magnitude of the axial force and, because of stability, on the slenderness of the member. No previous investigator has taken these factors into account. As it turns out, the weight-moment relationship is altered significantly for the beam-column with relatively large axial compression. This effect is taken into account in the author's method.

Instability of the frame which is unbraced (i.e., free to deflect laterally under gravity load) develops at loads which may be considerably smaller than those for the braced frame. No completely satisfactory practicable design procedure for treating this complex inelastic stability problem has been developed. An interim approximation, which is believed to be conservative for frames of proportions likely to be found in practice, has been proposed[5] and adopted in the American Institute of Steel Construction specifications for plastic design.[22] Example 2 shows how this interim provision can be taken into account in the optimization of weight for the unbraced frame. Although the initial solution does not coincide with the minimum-weight solution which was found by trial modifications of the initial solution, the latter is only one section removed in the table of standard sections (Table 1) from the former for both beam and columns.

Examples 3, 4, and 5 illustrate optimization of frames by means of a program written for the IBM 7090 computer. This program determines the best-fit straight-line weight-moment relationship for each member of the frame, based on an appropriate range of moment capacity, and obtains an initial solution by the simplex method of linear programming. The trial sequence of the program investigates variations in the initial solution to allow for the fact that even the best-fit straight-line weight-moment relationship for a limited range of standard sections results in some scatter, and, more importantly, the fact that a finite number, rather than a continuous spectrum of sections, exists.

Example 3 illustrates the solution of a two-bay, one-story frame, subjected to sidesway, with the attendant array of linear restrictions. The increase in size of the array over that of Example 2 should be noted. The initial solution is found to be the minimum-weight solution.

In the solution of Example 4, only one member of the minimum-weight frame differed from the corresponding member of the initial solution, and the two were adjacent in Table 1. Because of symmetry of frame and loading, there were only six inequalities to be satisfied. Thus, the solution was based on the complete array, so that it was unnecessary to check the yield criterion.

The particular reduced array of linear restrictions used in Example 5 did not produce the correct theoretical moments because one of the equations which is part of the basis of solution was omitted. Two different ways of completing the solution, both of which yielded satisfactory results, were then carried out. This example demonstrates the feasibility of solutions based upon reduced or incomplete arrays of restrictions.

Examples 4 and 5 also illustrate another aspect of minimum-weight design which was discussed in Section III G, namely, design for two or more load systems. The minimum-weight designs for gravity loads (Example 4), and for the combination of gravity load with wind (Example 5) satisfy Equation (3.20). Therefore, the minimum-weight structure is that of Example 4.

In all the examples, the members of the minimum-weight frame are either identical with or only one section removed (in the table of economy sections) from those of the initial solution. Of course, situations may arise in which the theoretical moments place some members of the initial frame two or even three sections from those of the minimum-weight frame. While this deviation is not serious for the small problem, the required computation time to examine all variations of the initial solution within even a bandwidth of, say, four for a larger problem may be prohibitive (Section III E). However, the author's experience with his method of solution points to some restrictions that may be placed on the bandwidth. In the first place, if the theoretical moment for any member, determined by the simplex solution, is found to lie at either the high or the low limit of the feasible range of moments, the bandwidth need be extended in only one direction. Secondly, in the cases enumerated below, bandwidths for particular members may be restricted even further.

(1) It will generally be found that, for the minimum-weight frame, moments in the interior columns lie at the lower limit of the feasible range of moments.

(2) In exterior columns of one-story frames braced against sidesway, moments for the minimum-weight solution will lie at the lower limit of the feasible range of moments if the ratio of column height to beam span is about 0.25 or greater, but at the upper limit of the range if the ratio is about 0.2 or smaller.

(3) Theoretical moments in exterior columns in the bottom story of a two-story frame braced against sidesway will generally lie at the lower limit of the feasible range of moments.

If the simplex solution places the theoretical moments at any of the limits indicated above, the designer should consider restricting the bandwidths in the trial solution for the members affected to a narrower width than is used for the remainder of the members in the structure. In fact, consideration should be given to a bandwidth of one, which amounts to holding these members to the sections of the initial solution.

Another possibility for the restriction of bandwidth occurs when the theoretical moment for any member falls close to the moment capacity of one of the more efficient sections. These more efficient sections are characterized by having a significant increase in M_p-to-weight ratio over those of adjacent economy sections of less weight. Reference to the partial list below will show this to be the case for the 14WF30, 16WF36, 18WF50, and 21WF62. It should be noted that each of these is the lightest economy section for its nominal depth.

Section	M_p/Weight	Per Cent Increase
21WF62	6.4	14.3
18WF60	5.6	0
18WF55	5.6	1.8
18WF50	5.5	10.0
16WF45	5.0	0
16WF40	5.0	2.0
16WF36	4.9	11.3
14WF34	4.4	2.3
14WF30	4.3	10.0
12WF27	3.9	--

When the theoretical moment for a particular member falls near the moment capacity of one of these sections, the minimum-weight solution is quite likely to include it. Thus, if the initial solution contains such a member, it will probably be safe to hold it constant in the subsequent exploration of variations in the initial design.

For members other than those discussed above, bandwidths should be set as wide as computational running-time limitations allow, if a high degree of precision in optimization is desired. However, it should be remembered that if bands are not wide enough to pick up the minimum-weight design, even the initial design will usually be only slightly heavier than the true minimum-weight design.

It was found that if the objective function was generated from best-fit straight lines (for the weight-moment relationships) whose average residuals were within about 15 per cent of a_k (Equation 3.3) or a_i' (Equation 3.9), the simplex solution generally yielded excellent results.

• • •

VI. CONCLUSIONS

The method of optimization developed in this report allows for the determination of the minimum-weight design of steel frames within the restrictions imposed in Section III B. The method includes the effects of axial loading, overall frame instability due to sidesway, and the nonlinear relationship between weight and moment capacity of standard sections.

Although standard sections are used in the frames of the examples, frames using built-up sections can also be optimized, provided that a linear weight-moment equation for the range of proposed built-up sections is determined. Furthermore, it should be noted that, with the built-up section, a virtually continuous spectrum of moment capacities exists, eliminating the need for the trial portion of the computer program.

Although gable and other nonorthogonal frames are not considered in the examples, they can be optimized by the method proposed and accommodated by the computer program which was developed.

If all the attendant independent linear restrictions are considered, a two-story, two-bay, unsymmetrical frame subject to sidesway represents about the upper limit to the problems that it is feasible to solve by this method on the 7090 computer. However, if a reduced array is used, and the minimum-weight solution is checked against the yield criterion, as was done in Examples 4 and 5, most building frames which the engineer is likely to meet can be accommodated.

• • •

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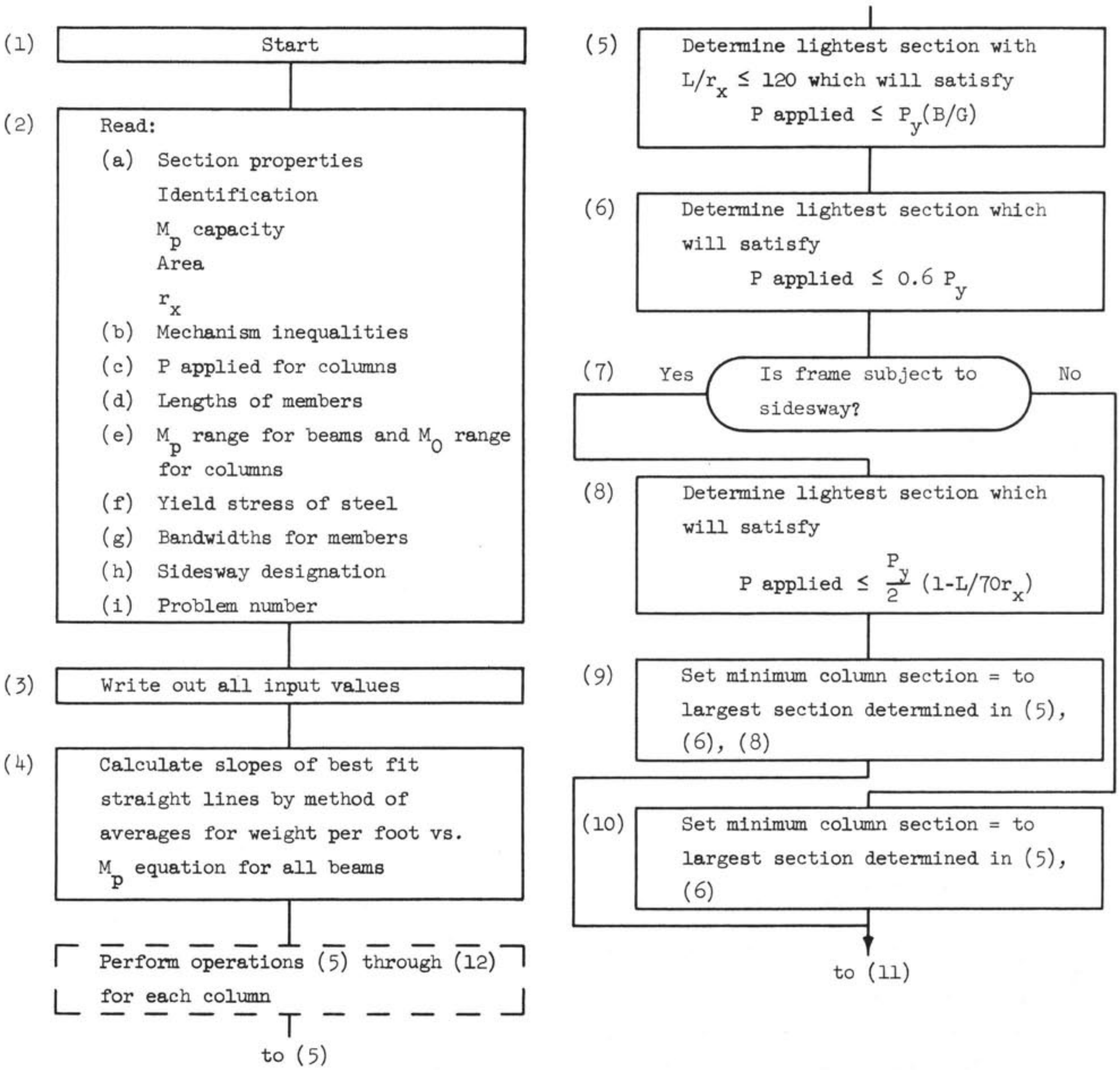
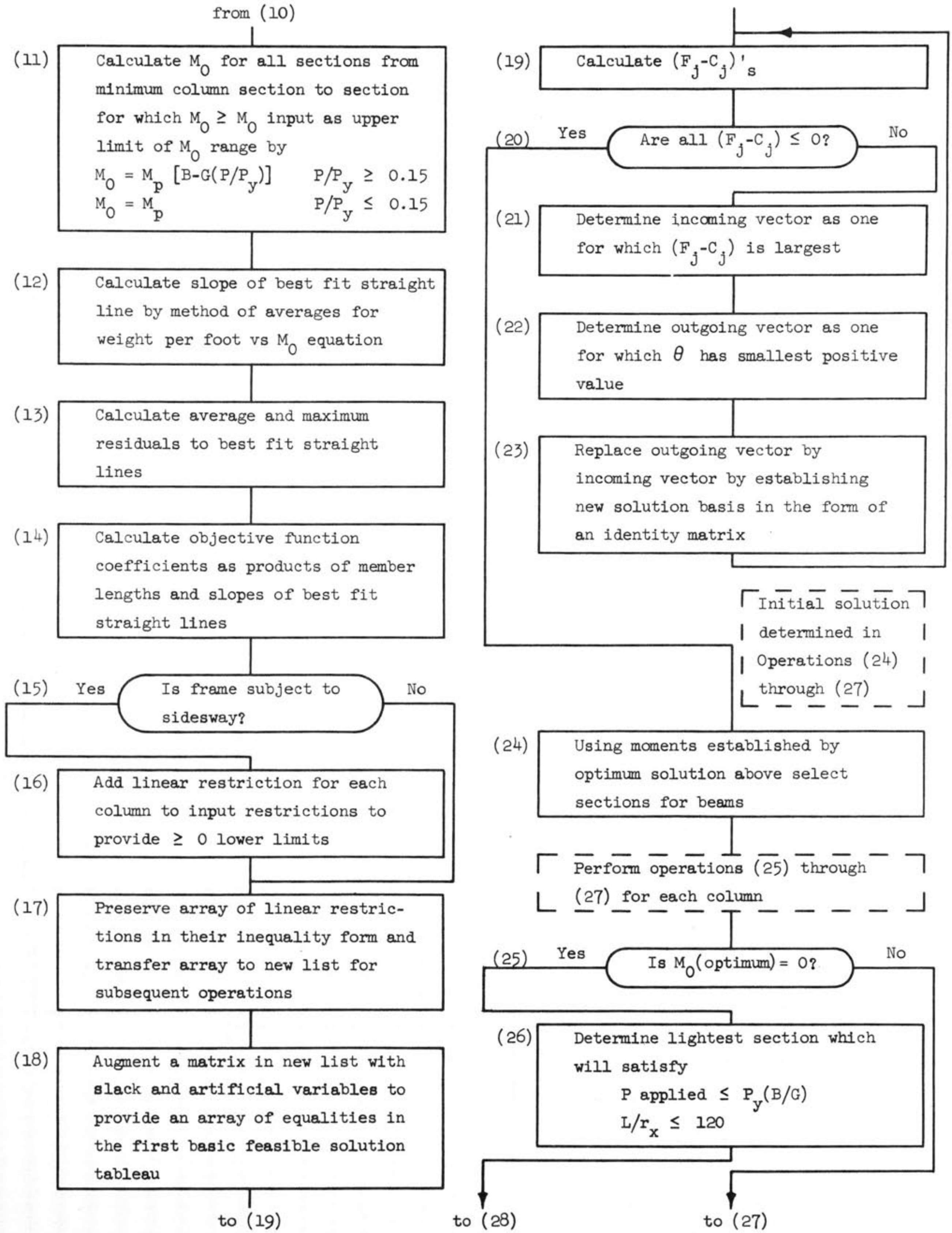
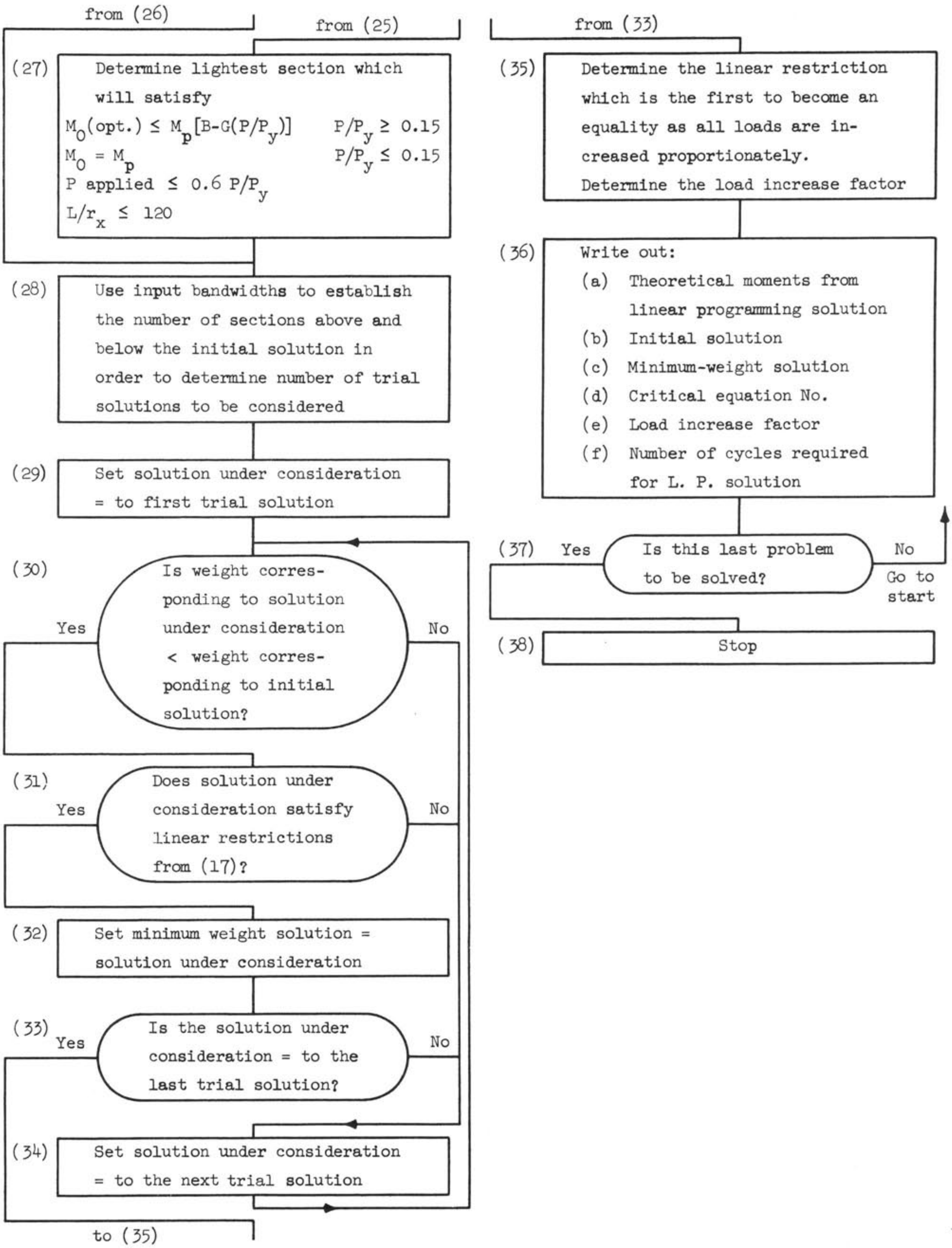
VIII. APPENDIX: FLOW DIAGRAM FOR MINIMUM-WEIGHT PROGRAM




from (10)
(11) Calculate M_0 for all sections from minimum column section to section for which $M_0 \geq M_0$ input as upper limit of M_0 range by
$M_0 = M_p\,[B-G(P/P_y)]$ $\quad P/P_y \geq 0.15$
$M_0 = M_p$ $\quad P/P_y \leq 0.15$
(12) Calculate slope of best fit straight line by method of averages for weight per foot vs M_0 equation
(13) Calculate average and maximum residuals to best fit straight lines
(14) Calculate objective function coefficients as products of member lengths and slopes of best fit straight lines
(15) Is frame subject to sidesway?
Yes
No
(16) Add linear restriction for each column to input restrictions to provide ≥ 0 lower limits
(17) Preserve array of linear restrictions in their inequality form and transfer array to new list for subsequent operations
(18) Augment a matrix in new list with slack and artificial variables to provide an array of equalities in the first basic feasible solution tableau
to (19)
(19) Calculate (F_j-C_j)'s
(20) Are all $(F_j-C_j) \leq 0$?
Yes
No
(21) Determine incoming vector as one for which (F_j-C_j) is largest
(22) Determine outgoing vector as one for which θ has smallest positive value
(23) Replace outgoing vector by incoming vector by establishing new solution basis in the form of an identity matrix
Initial solution determined in Operations (24) through (27)
(24) Using moments established by optimum solution above select sections for beams
Perform operations (25) through (27) for each column
(25) Is M_0(optimum) = 0?
Yes
No
(26) Determine lightest section which will satisfy
P applied $\leq P_y(B/G)$
$L/r_x \leq 120$
to (28)
to (27)


from (26)
from (25)
(27)
Determine lightest section which will satisfy
M_0(opt.) ≤ M_p[B-G(P/P_y)] P/P_y ≥ 0.15
M_0 = M_p P/P_y ≤ 0.15
P applied ≤ 0.6 P/P_y
L/r_x ≤ 120
(28)
Use input bandwidths to establish the number of sections above and below the initial solution in order to determine number of trial solutions to be considered
(29)
Set solution under consideration = to first trial solution
(30)
Is weight corresponding to solution under consideration < weight corresponding to initial solution?
Yes
No
(31)
Does solution under consideration satisfy linear restrictions from (17)?
Yes
No
(32)
Set minimum weight solution = solution under consideration
(33)
Is the solution under consideration = to the last trial solution?
Yes
No
(34)
Set solution under consideration = to the next trial solution
to (35)
from (33)
(35)
Determine the linear restriction which is the first to become an equality as all loads are increased proportionately.
Determine the load increase factor
(36)
Write out:
(a) Theoretical moments from linear programming solution
(b) Initial solution
(c) Minimum-weight solution
(d) Critical equation No.
(e) Load increase factor
(f) Number of cycles required for L. P. solution
(37)
Yes
Is this last problem to be solved?
No
Go to start
(38)
Stop

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